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*Piotr Błaszczyk, Joanna Major*

# Calculus without the concept of limit*

**Abstract.** There are two different approaches to nonstandard analysis: semantic (model-theoretic) and syntactic (axiomatic). Both of these approaches require some knowledge of mathematical logic. We present a method based on an ultrapower construction which does not require any mathematical logic prerequisites. On the one hand, it is a complementary course to a standard calculus course. On the other hand, since it relies on a different intuitive background, it provides an alternative approach. While in standard analysis an intuition of *being close* is represented by the notion of limit, in nonstandard analysis it finds its expression in the relation *is infinitely close*. As a result, while standard courses focus on the $\varepsilon - \delta$ technique, we explore an algebra of infinitesimals. In this paper, we offer a proof of the theorem on the equivalency of limits and infinitesimals, showing that calculus can be developed without the concept of limit.

## Introduction

Nonstandard analysis was developed by Abraham Robinson in the early 60s of the past century. It applied sophisticated techniques of model theory and required a deep logical background. Then, in the late 60s, Elias Zakon developed a sematic approach using set-theoretic objects called superstructures. It extends a superstructure $V(S)$ on the set $S$ to another object $V(S)^*$ using an ultrapower construction together with a mapping $V(S) \mapsto V(S)^*$ which satisfies the so-called transfer principle. The transfer principle relates the formal properties of the starting structure to the extended one. This approach became more suitable for use by mathematicians who are non-specialists in model theory, however, due to the transfer principle, it still includes some parts of mathematical logic.

In the mid-1970s, Edward Nelson introduced an axiomatic formulation of nonstandard analysis. Nelson's system is an extension of Zermelo-Fraenkel set theory. Although his approach requires less logic, it needs some foundational skills: the axiom of comprehension, which mathematicians usually take for granted, needs to be applied with great care.

*Analiza matematyczna bez pojęcia granicy

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To this day, many new different axiomatic formulations of nonstandard analysis have appeared (see Kanovei, Reeken, 2004).

Right from its very beginning, the didactic role of nonstandard analysis was examined. Jerome Keisler developed an approach based on Robinson's nonstandard analysis and started using infinitesimals in beginning U.S. calculus courses in 1969. His book (Keisler, 1976) was widely used in the 1970s as an alternative approach for teaching differential and integral calculus. In the 1990s, Nelson's development of nonstandard analysis found its didactic implementation in France (see Deledicq, 1995). And as soon as a new axiomatic formulation of nonstandard analysis appears, it finds its didactic counterpart (see O'Donovan, 2007).

In this paper we present a development of nonstandard analysis based on an ultrapower construction which does not require any mathematical logic prerequisites. On the one hand, it can be considered as a complementary course to a standard calculus course. On the other hand, since it relies on a different intuitive background, it provides an alternative approach. While in standard analysis an intuition of *being close* is represented by the notion of limit, in nonstandard analysis it finds its expression in the relation *is infinitely close*, $x \approx y$. As a result, while standard courses focus on the $\varepsilon - \delta$ technique, we explore an algebra of infinitesimals. In what follows, we offer a proof of the theorem on the equivalency of limits and infinitesimals, namely: If $(r_n)$ is a sequence of real numbers and $g \in \mathbb{R}$, then

$$\lim_{n\to\infty} r_n = g \Leftrightarrow (\forall K \in \mathbb{N}_\infty)(r^*_K \approx g^*).$$

Our development is self-contained, including all necessary definitions and proofs. A similar attitude can be found in (Lindstrøm, 1988), however, our approach is didactics motivated and explores the properties of an ordered field. Moreover, while the above mentioned theorem makes the crux of our development, in (Lindstrøm, 1988) it is omitted.

## 1. Ordered fields

In this section, we restate the definition of the field of real numbers. Both reals and nonstandard reals share the properties of ordered fields, thus, for the sake of completeness, we start with the definition of a totally ordered field (ordered field, for short).

Definition 1
A binary relation $<$ on a set $\mathbb{F}$ is a total order if it is transitive and for every $x, y \in \mathbb{F}$ exactly one of the following conditions holds

$$x < y,\ x = y,\ x > y.$$

Definition 2
An ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ is a commutative field together with a total order that is compatible with addition and multiplication,

$$(\forall x, y \in \mathbb{F})(x < y \Rightarrow x + z < y + z),$$

$(\forall x, y, z \in \mathbb{F})(x < y, 0 < z \Rightarrow xz < yz)$.

From now on, let $\mathcal{F}$ denote an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$; by $\mathbb{F}_+$ we denote the set of positive elements $\{x \in \mathbb{F} : x > 0\}$.

The field of fractions $(\mathbb{Q}, +, \cdot, 0, 1, <)$ is the *smallest* ordered field in the sense that it can be embedded into every ordered field (see Cohen, Ehrlich, 1963, p. 68). Thus, symbols such as $\frac{n}{m}$ make sense in any ordered field.

In any ordered field, the absolute value function is defined by

$$|x| = \begin{cases} x, \text{ if } x > 0, \\ 0, \text{ if } x = 0, \\ -x, \text{ if } x < 0. \end{cases} \tag{1}$$

This function obeys the well-known laws:

$$|x| = |-x|, \quad |x+y| \leqslant |x| + |y|, \quad |xy| = |x||y|, \quad ||x| - |y|| \leqslant |x-y|.$$

Showing that nonstandard reals form an ordered field, we make use of the definition and the properties of the absolute value.

Note that the notion of limit of a sequence $(a_n) \subset \mathbb{F}$ given by

$$\lim_{n\to\infty} a_n = g \Leftrightarrow (\forall \varepsilon \in \mathbb{F}_+)(\exists k \in \mathbb{N})(\forall n \in \mathbb{N})(n > k \Rightarrow |a_n - g| < \varepsilon),$$

exploits the absolute value and makes sense in any ordered field. Likewise, the algebra of limits, i.e., laws such as

$$\text{if } \lim_{n\to\infty} a_n = g, \ \lim_{n\to\infty} b_n = h, \text{ then } \lim_{n\to\infty}(a_n + b_n) = g + h,$$

can be developed in any ordered field (see Cohen, Ehrlich, 1963, p. 73–79).

## 1.1. The field of real numbers

We define real numbers to be a commutative field together with a continuous order. However, the continuity of a total order can take many forms. In this paper, we apply what we believe to be the simplest definition – the one introduced by Richard Dedekind in his 1872 (Dedekind, 1872). To this end, we need a notion of Dedekind cut.

Definition 3

A pair of sets $(L, U)$ is a Dedekind cut of a totally ordered set $(X, <)$ if (1) $L, U \neq \emptyset$, (2) $L \cup U = X$, (3) $(\forall y \in L)(\forall z \in U)(y < z)$. A cut $(L, U)$ is called a gap if there exists neither a maximum in $L$ nor a minimum in $U$.

Definition 4

The field of real numbers is an ordered field $\mathcal{F}$, in which every Dedekind cut $(L, U)$ of $(\mathbb{F}, <)$ satisfies the condition

$$(\exists z \in \mathbb{F})(\forall x \in L)(\forall y \in U)(x \leqslant z \leqslant y). \tag{C1}$$

The categoricity theorem states that every two ordered fields satisfying axiom (C1) are isomorphic (Cohen, Ehrlich, 1963, p. 105). In other words, every ordered field satisfying (C1) is isomorphic to the field of real numbers $(\mathbb{R}, +, \cdot, 0, 1, <)$.

The order of an ordered field $\mathcal{F}$ is dense. Indeed, if $x, y \in \mathbb{F}$ and $x < y$, then $x < \frac{1}{2}(x+y) < y$. It follows from the density, that there exists exactly one element guaranteed by (C1), therefore, we also use this form of the continuity axiom

$$(\exists! z \in \mathbb{F})(\forall x \in L)(\forall y \in U)(x \leqslant z \leqslant y).$$

## 1.2. Archimedean field

Definition 5
An ordered field $\mathcal{F}$ is Archimedean if it satisfies the condition

$$(\forall x, y \in \mathbb{F})(\exists n \in \mathbb{N})(0 < x < y \Rightarrow nx > y). \tag{A1}$$

Axiom (A1) is called the Archimedean axiom. Following are its equivalent versions

(A2) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$,

(A3) $(\forall x \in \mathbb{F})(\exists n \in \mathbb{N})(n > x)$,

(A4) $(\forall x, y \in \mathbb{F})(\exists q \in \mathbb{Q})(x < y \Rightarrow x < q < y)$.

(A5) If $(L, U)$ is a Dedekind cut of $(\mathbb{F}, <)$, then

$$(\forall n \in \mathbb{N})(\exists x \in L)(\exists y \in U)(y - x < 1/n).$$

Given $(L, U)$ is a Dedekind cut of $(\mathbb{F}, <)$, by (A4), we can find such a sequence $(r_n) \subset \mathbb{F}$ that satisfies conditions

$$r_{2k-1} \in L,\ r_{2k} \in U,\ r_{2k} - r_{2k-1} < 1/k,$$

$$r_1 \leqslant r_3 \leqslant ... \leqslant r_{2k-1} \leqslant ... \leqslant r_{2k} \leqslant ... \leqslant r_4 \leqslant r_2.$$

We refer to this fact proving Theorem 4 below.

Theorem 1
*The field of real numbers is an Archimedean field.*

*Proof.* Seeking a contradiction, suppose that there exist $r, s \in \mathbb{R}$ such that $0 < r < s$ and for every $n \in \mathbb{N}$ inequality $nr \leqslant s$ holds. Set

$$L = \{x \in \mathbb{R} : (\exists n \in \mathbb{N})(x \leqslant nr)\}, \quad U = \{x \in \mathbb{R} : (\forall n \in \mathbb{N})(x > nr)\}.$$

The pair $(L, U)$ is a Dedekind cut of the line $(\mathbb{R}, <)$. Next, by (C1), there exists exactly one $\gamma$ such that

$$(\forall x \in L)(\forall y \in U)(x \leqslant \gamma \leqslant y).$$

Moreover, for every $n \in \mathbb{N}$, numbers $nr$ belong to $L$ and inequalities $nr \leqslant \gamma$ hold. Now, the number $\gamma - r$ belongs to $L$, hence $\gamma - r \leqslant nr$, for some $n$. It follows that $\gamma \leqslant (n+1)r$, and as a result, $\gamma < (n+2)r$. Since $(n+2)r \in L$, this contradicts the assumption that $\gamma$ is not less than any element of $L$.

The field of real numbers is the *biggest* Archimedean field, that is, any Archimedean field can be embedded into the field of reals. Thus, any extension of the reals is a non-Archimedean field (see Cohen, Ehrlich, p. 87–88; Hartshorne, p. 139). We will refer to this claim in Theorem 3 below in order to show that the nonstandard reals form a non-Archimedean field.

## 2. Ultrapower

### 2.1. Ultrafilter

In this section, we introduce the construction called the ultrapower. We use it to extend the field of reals to the field of nonstandard reals. We start with the definition of an ultrafilter, since it is a key concept in this construction, and present some basic results concerning ultrafilters.

Definition 6
A family of sets $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ is an ultrafilter on $\mathbb{N}$ if (1) $\emptyset \notin \mathcal{U}$, (2) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$, (3) if $A \in \mathcal{U}$ and $A \subset B$, then $B \in \mathcal{U}$, (4) for each $A \subset \mathbb{N}$, either $A$ or its complement $\mathbb{N} \setminus A$ belongs to $\mathcal{U}$.

Take the family of sets with finite complements,

$$\{A \subset \mathbb{N} : \mathbb{N} \setminus A \;\text{ is finite}\}.$$

This family is usually called a Fréchet filter on $\mathbb{N}$. Indeed, it obviously satisfies conditions (1)–(3) listed in the Definition 6. First, since the set $\mathbb{N} \setminus \emptyset$ is infinite, $\emptyset \notin \mathcal{U}$. Secondly, if the sets $\mathbb{N} \setminus A$ and $\mathbb{N} \setminus B$ are finite, then $\mathbb{N} \setminus A \cap B$ is also finite, for $\mathbb{N} \setminus A \cap B = \mathbb{N} \setminus A \cup \mathbb{N} \setminus B$. Thirdly, if the set $\mathbb{N} \setminus A$ is finite and the inclusion $A \subset B$ holds, then the set $\mathbb{N} \setminus B$ is also finite, for $\mathbb{N} \setminus B \subset \mathbb{N} \setminus A$.

Note, however, that neither the set of odds numbers nor the set of even numbers has a finite complement, hence, the Fréchet filter is not an ultrafilter. Although, by Zorn's lemma, it can be extended to an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ (see Cohen, 1991, p. 29; Goldblatt, 1998, p. 20–21; Robinson, 1966 p. 15-17).

From now on, $\mathcal{U}$ denotes a fixed ultrafilter on $\mathbb{N}$ containing every subset with a finite complement[1].

Following are obvious consequences of the above definitions:

1. $\mathbb{N} \in \mathcal{U}$;
2. for any $n \in \mathbb{N}$, if $A_i \in \mathcal{U}$, $1 \leqslant i \leqslant n$, then $\bigcap_{i=1}^{n} A_i \in \mathcal{U}$;
3. if $A \subset \mathbb{N}$, then either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$, and it is not the case that both $A$ and $\mathbb{N} \setminus A$ belong to $\mathcal{U}$;
4. if $A$ is a finite subset of $\mathbb{N}$, then $\mathbb{N} \setminus A \in \mathcal{U}$, in particular, we obtain $\mathbb{N} \setminus \{1, ..., n\} \in \mathcal{U}$, for every $n \in \mathbb{N}$[2].

[1] The reader can find the same reasoning in the literature encoded into one short sentence: Let $\mathcal{U}$ be a fixed nonprincipal ultrafiltr on $\mathbb{N}$.

[2] We assume that the set $\mathbb{N}$ does not include 0.

In the proofs provided below, we apply yet another characteristic of the ultrafilter $\mathcal{U}$, namely

THEOREM 2
*Let* $\{A_1, ..., A_n\}$ *be a finite family of pairwise disjoint sets,* $A_i \cap A_j = \emptyset$, *for* $1 \leqslant i < j \leqslant n$, *such that* $\bigcup_{i=1}^n A_i \in \mathcal{U}$. *Then* $A_i \in \mathcal{U}$ *for exactly one* $i$ *such that* $1 \leqslant i \leqslant n$.

**Proof** Let $\bigcup_{i=1}^n A_i \in \mathcal{U}$. For the first part of the proof, suppose that none of the sets $A_i$ belongs to $\mathcal{U}$. Since $\mathcal{U}$ is the ultrafilter, complements of these sets, $\mathbb{N} \setminus A_i$ belong to $\mathcal{U}$, for $1 \leqslant i \leqslant n$. As a result $\bigcap_{i=1}^n (\mathbb{N} \setminus A_i) \in \mathcal{U}$. On the other hand, it follows from the equality

$$\bigcap_{i=1}^{n} (\mathbb{N} \setminus A_i) = \mathbb{N} \setminus \bigcup_{i=1}^{n} A_i,$$

that $\mathbb{N} \setminus \bigcup_{i=1}^n A_i \in \mathcal{U}$. This contradicts $\bigcup_{i=1}^n A_i \in \mathcal{U}$.

Now, suppose that for some different indexes $i, j$ obtains $A_i, A_j \in \mathcal{U}$. Then, $A_i \cap A_j \in \mathcal{U}$. Since $A_i, A_j$ are disjoint sets, it follows that the empty set belongs to $\mathcal{U}$. This contradicts the very definition of the ultrafilter $\mathcal{U}$.

## 2.2. Ultrapower

In the product $\mathbb{R}^{\mathbb{N}}$ we define a relation

$$(r_n) \equiv (s_n) \Leftrightarrow \{n \in \mathbb{N} : \ r_n = s_n\} \in \mathcal{U}.$$

This is easily seen to be a relation satisfying reflexivity and symmetry. Indeed, since

$$\{n \in \mathbb{N} : \ r_n = r_n\} = \mathbb{N} \text{ and } \{n \in \mathbb{N} : \ r_n = s_n\} = \{n \in \mathbb{N} : \ s_n = r_n\},$$

relations obtain: first, $(r_n) \equiv (r_n)$, second, if $(r_n) \equiv (s_n)$, then $(s_n) \equiv (r_n)$.

As for transitivity, note that, if $(r_n) \equiv (s_n)$ and $(s_n) \equiv (t_n)$, then

$$\{n \in \mathbb{N} : \ r_n = s_n\} \in \mathcal{U}, \ \{n \in \mathbb{N} : \ s_n = t_n\} \in \mathcal{U}.$$

Hence

$$\{n \in \mathbb{N} : \ r_n = s_n\} \cap \{n \in \mathbb{N} : \ s_n = t_n\} \in \mathcal{U}.$$

Therefore, since the following inclusion holds

$$\{n \in \mathbb{N} : \ r_n = s_n\} \cap \{n \in \mathbb{N} : \ s_n = t_n\} \subseteq \{n \in \mathbb{N} : \ r_n = t_n\},$$

the set $\{n \in \mathbb{N} : \ r_n = t_n\}$ belongs to $\mathcal{U}$ and we obtain $(r_n) \equiv (t_n)$.

Finally, $\equiv$ is an equivalence relation.

Let $\mathbb{R}^*$ denote the reduced product $\mathbb{R}^{\mathbb{N}}/_{\equiv}$. We call the elements of the set $\mathbb{R}^*$ nonstandard real numbers, or hyperreals, for short.

The equality relation in $\mathbb{R}^*$ is obviously given by

$$[(r_n)] = [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : \; r_n = s_n\} \in \mathcal{U}. \tag{2}$$

It follows from the notion of ultrafilter that

$$[(r_n)] \neq [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : \; r_n \neq s_n\} \in \mathcal{U}. \tag{3}$$

Indeed, we start with the equivalence

$$[(r_n)] \neq [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : \; r_n = s_n\} \notin \mathcal{U}.$$

If $\{n \in \mathbb{N} : \; r_n = s_n\} \notin \mathcal{U}$, then the set $\mathbb{N} \setminus \{n \in \mathbb{N} : \; r_n = s_n\}$ belongs to $\mathcal{U}$. By the equality

$$\mathbb{N} \setminus \{n \in \mathbb{N} : \; r_n = s_n\} = \{n \in \mathbb{N} : \; r_n \neq s_n\},$$

we obtain (3).

Algebraic operations on $\mathbb{R}^*$ are defined pointwise, that is

$$[(r_n)] + [(s_n)] = [(r_n + s_n)], \quad [(r_n)] \cdot [(s_n)] = [(r_n \cdot s_n)]. \tag{4}$$

A total order on $\mathbb{R}^*$ is given by the following definition

$$[(r_n)] < [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{U}^3. \tag{5}$$

We embed the set of reals $\mathbb{R}$ into the set of hyperreals $\mathbb{R}^*$ by identifying the standard real number $r$ with the hyperreal determined by the constant sequence $(r, r, r, ...)$. Thus, setting

$$r^* = [(r, r, r, ...)], \tag{6}$$

the embedding is given by the map $\mathbb{R} \ni r \mapsto r^* \in \mathbb{R}^*$.

We should yet show that the above definitions are correct, that is:
if $(r_n) \equiv (r'_n)$ and $(s_n) \equiv (s'_n)$, then

$$[(r_n)] = [(s_n)] \Leftrightarrow [(r'_n)] = [(s'_n)],$$
$$[(r_n)] + [(s_n)] = [(r'_n)] + [(s'_n)], \quad [(r_n)] \cdot [(s_n)] = [(r'_n)] \cdot [(s'_n)],$$
$$[(r_n)] < [(s_n)] \Leftrightarrow [(r'_n)] < [(s'_n)].$$

The justification is the same as the one given above in terms of the relation $\equiv$. We use some new tricks that apply the basic properties of ultrafilter proving Theorem 3 below.

[3] In the above definitions, we adopt a standard convention to use the same signs for the relations and operations on $\mathbb{R}$ and $\mathbb{R}^*$.

## 3. The field of hyperreals

Standard developments of nonstandard analysis make use of the so-called transfer principle (see Bair, Błaszczyk, Ely, Henry, Kanovei, Katz, Katz, Kutateladze, McGaffey, Sherry, Shnider, 2013, § 4.5) or just Łoś Theorem (1955) to show that hyperreals form an ordered field. In what follows, we make use only of the properties of the ultrafilter $\mathcal{U}$ detailed in § 2.1 and the definitions given in § 2.2.

Theorem 3
*The structure of hyperreals* $(\mathbb{R}^*, +, \cdot, 0^*, 1^*, <)$ *is a non-Archimedean ordered field.*

*Proof.* 1. In the first part, we show that the structure $(\mathbb{R}^*, +, \cdot, 0^*, 1^*)$ is a commutative field.

It follows from the commutativity of addition of standard reals that the equality holds

$$\{n \in \mathbb{N} : r_n + s_n = s_n + r_n\} = \mathbb{N}.$$

Hence, by (2), we obtain $[(r_n + s_n)] = [(s_n + r_n)]$, which means by (4), that

$$[(r_n)] + [(s_n)] = [(s_n)] + [(r_n)].$$

Similarly, it follows from the commutativity of multiplication, the associativity of addition and multiplication, and the distributive law of standard reals that the sets listed below all are equal to $\mathbb{N}$,

$$\begin{aligned}
&\{n \in \mathbb{N} : r_n \cdot s_n = s_n \cdot r_n\},\\
&\{n \in \mathbb{N} : (r_n + s_n) + t_n = s_n + (r_n + t_n)\},\\
&\{n \in \mathbb{N} : (r_n \cdot s_n) \cdot t_n = s_n \cdot (r_n \cdot t_n)\},\\
&\{n \in \mathbb{N} : (r_n + s_n) \cdot t_n = r_n \cdot t_n + s_n \cdot t_n\}.
\end{aligned}$$

Hence, by (2), we obtain $[(r_n \cdot s_n)] = [(s_n \cdot r_n)]$ etc., and by (4), $[(r_n)] \cdot [(s_n)] = [(s_n)] \cdot [(r_n)]$ etc.

The additive inverse of $[(r_n)]$ is given by

$$-[(r_n)] = [(-r_n)]. \tag{7}$$

In fact, it easily follows from (2) and (6) that the equalities hold

$$[(r_n)] + [(-r_n)] = [(0, 0, 0, ...)] = 0^*.$$

In order to show that there exists a multiplicative inverse, suppose that $[(r_n)] \neq 0^*$. Then, by (3), we obtain $\{n \in \mathbb{N} : r_n \neq 0\} \in \mathcal{U}$. Set $J = \{n \in \mathbb{N} : r_n \neq 0\}$ and define an inverse of $[(r_n)]$ by putting

$$[(r_n)]^{-1} = [(a_n)],$$

where

$$a_n = \begin{cases} r_n^{-1}, \text{ if } n \in J, \\ \quad 1, \text{ if } n \notin J. \end{cases} \tag{8}$$

Now, note that $J = \{n \in \mathbb{N} : a_n \cdot r_n = 1\}$. Since $J \in \mathcal{U}$, by (2), we obtain

$$[(a_n \cdot r_n)] = 1^*.$$

2. In this part, we deal with the order of hyperreals. First, observe that for every $[(r_n)], [(s_n)] \in \mathbb{R}^*$, by Definition 1, we obtain the following equality

$$\{n \in \mathbb{N} : r_n < s_n\} \cup \{n \in \mathbb{N} : r_n = s_n\} \cup \{n \in \mathbb{N} : r_n > s_n\} = \mathbb{N}.$$

Since $\mathbb{N} \in \mathcal{U}$ and the sets

$$\{n \in \mathbb{N} : r_n < s_n\},\ \{n \in \mathbb{N} : r_n = s_n\},\ \{n \in \mathbb{N} : r_n > s_n\}$$

are pairwise disjoined, by Theorem 2, exactly of one them belongs to $\mathcal{U}$. Thus, exactly one of the following conditions holds

$$[(r_n)] < [(s_n)],\ [(r_n)] = [(s_n)],\ [(r_n)] > [(s_n)].$$

Secondly, if $[(r_n)] < [(s_n)]$ and $[(s_n)] < [(t_n)]$, then the sets

$$\{n \in \mathbb{N} : r_n < s_n\},\ \{n \in \mathbb{N} : s_n < t_n\},\ \{n \in \mathbb{N} : r_n < s_n\} \cap \{n \in \mathbb{N} : s_n < t_n\}$$

belong to $\mathcal{U}$. Since

$$\{n \in \mathbb{N} : r_n < s_n\} \cap \{n \in \mathbb{N} : s_n < t_n\} \subseteq \{n \in \mathbb{N} : r_n < t_n\},$$

we obtain $\{n \in \mathbb{N} : r_n < t_n\} \in \mathcal{U}$. Finally, $[(r_n)] < [(t_n)]$.

Thirdly, we proceed in a similar way to show that the order of hyperreals is compatible with multiplication. Set

$$J = \{n \in \mathbb{N} : r_n < s_n\},\ K = \{n \in \mathbb{N} : 0 < t_n\},\ L = \{n \in \mathbb{N} : r_n \cdot t_n < s_n \cdot t_n\}.$$

If $J, K \in \mathcal{U}$, then $J \cap K \in \mathcal{U}$. Next, since the order of the reals is compatible with multiplication, we have $J \cap K \subset L$.

Now, if $[(r_n)] < [(s_n)]$ and $0^* < [(t_n)]$, then $J \cap K \in \mathcal{U}$, and $L \in \mathcal{U}$. Finally we obtain

$$[(r_n)] \cdot [(t_n)] < [(s_n)] \cdot [(t_n)].$$

In the same way we show that the order of hyperreals is compatible with addition.

3. In this final part, we show that the hyperreals do not obey the Archimedean axiom. To this end, first observe that

$$\{n \in \mathbb{N} : 0 < n^{-1}\} = \mathbb{N},$$

which means that the inequality $0^* < [(1/n)]$ holds. Then, by (A2), for every positive real number $r$ there exists $k \in \mathbb{N}$ such that $1/n < r$, for $n > k$. Therefore,

$$\{k + 1, k + 2, ...\} \subseteq \{n \in \mathbb{N} : n^{-1} < r\}.$$

Since the set $\{n \in \mathbb{N} : k+1 \leqslant n\}$ belongs to $\mathcal{U}$, the set $\{n \in \mathbb{N} : n^{-1} < r\}$ also belongs to $\mathcal{U}$. As a consequence, the inequality obtains $[(1/n)] < r^*$. Finally, we have

$$0^* < [(1/n)] < r^*, \quad \text{for every } r \in \mathbb{R}_+.$$

This means that the field of hyperreals extends the field of reals. Thus, the field of hyperreals is a non-Archimedean field.

Remark 1

It is worth noting, that defining a hypperreal $[(t_n)]$ is enough to determine terms $t_n$ only for indexes $n$ that belong to some element of $\mathcal{U}$. To illustrate this phenomenon, let $[(a_n)]$ be the inverse of the hyperreal $[(r_n)]$ given by the definition (8), let $J \in \mathcal{U}$, and set

$$b_n = \begin{cases} r_n^{-1}, \text{ if } n \in J, \\ 0, \text{ if } n \notin J. \end{cases}$$

Since $J = \{n \in \mathbb{N} : a_n = b_n\}$, by (2), the equality $[(a_n)] = [(b_n)]$ holds. However, we can go even further and take

$$c_n = \begin{cases} r_n^{-1}, \text{ if } n \in J, \\ x, \text{ if } n \notin J, \end{cases}$$

where $x$ just means some real number. Since $J \subseteq \{n \in \mathbb{N} : a_n = c_n\}$, the set $\{n \in \mathbb{N} : a_n = c_n\}$ belongs to $\mathcal{U}$ and the equality $[(a_n)] = [(c_n)]$ still holds.

To sum up, the inverse $[(c_n)]$ of the hyperreal $[(r_n)]$ can be defined by putting

$$c_n = r_n^{-1}, \quad \text{if } n \in K, \text{ for some } K \in \mathcal{U}. \tag{9}$$

Remark 2

The set of positive hyperintegers (hypernaturals) $\mathbb{N}^*$ and the set of hyperfractions $\mathbb{Q}^*$ are defined by

$$\mathbb{N}^* = \{[(r_n)] \in \mathbb{R}^* : \ \{n \in \mathbb{N} : \ r_n \in \mathbb{N}\} \in \mathcal{U}\},$$

$$\mathbb{Q}^* = \{[(r_n)] \in \mathbb{R}^* : \ \{n \in \mathbb{N} : \ r_n \in \mathbb{Q}\} \in \mathcal{U}\}.$$

Upon the previous remark, we can assume that

$$[(r_n)] \in \mathbb{N}^* \Leftrightarrow r_n \in \mathbb{N}, \text{ for every } n \in \mathbb{N},$$

$$[(r_n)] \in \mathbb{Q}^* \Leftrightarrow r_n \in \mathbb{Q}, \text{ for every } n \in \mathbb{N}.$$

This is why, from now on, $[(n_j)]$, $[(q_n)]$ denote a hyperinteger and hyperfraction respectively. With these definitions in mind, we return to the Archimedean axiom.

While the field of hyperrelas is non-Archimedean, it still obeys the following transformations of the Archimedean axiom:

(A3*) $(\forall x \in \mathbb{R}^*)(\exists n \in \mathbb{N}^*)(n > x)$,

(A4*) $(\forall x, y \in \mathbb{R}^*)(\exists q \in \mathbb{Q}^*)(x < y \Rightarrow x < q < y)$.

To show (A3$^*$), let $[(r_j)]$ be a hyperreal. By (A3), there exists $n_j > r_j$, for every $j \in \mathbb{N}$. Set $K = [(n_j)]$. Obviously, $K \in \mathbb{N}^*$, moreover,

$$\{j \in \mathbb{N} : n_j > r_j\} = \mathbb{N},$$

which means that $K > [(r_j)]$.

Likewise, let $[(r_j)]$, $[(s_j)]$ be hyperreals such that $[(r_j)] < [(s_j)]$. Then, the set $J = \{j \in \mathbb{N} : r_j < s_j\}$ belongs to $\mathcal{U}$. By (A4), there exists $q_j$ such that $r_j < q_j < s_j$, for every $j \in J$. Although $q_j$ is not determined for every $j \in \mathbb{N}$, by the previous remark, we can claim that $[(q_j)] \in \mathbb{Q}$.

Now, since

$$r_j < q_j < s_j, \text{ for every } j \in J,$$

we obtain

$$[(r_j)] < [(q_j)] < [(s_j)].$$

Remark 3
Since hyperreals form an ordered field, the absolute value function on the set $\mathbb{R}^*$ is defined in the standard way (see § 1 above).

Lemma 1
*The relationship between the absolute values on* $\mathbb{R}$ *and* $\mathbb{R}^*$ *is given by the following equality*

$$\big|[(r_n)]\big| = [(|r_n|)].$$

*Proof.* We have to consider three cases. First, suppose that $[(r_n)] > 0^*$. By (1), it follows that $\big|[(r_n)]\big| = [(r_n)]$. Then, by Definition (6), the set $\{n \in \mathbb{N} : r_n > 0\}$ belongs to $\mathcal{U}$. Again, by (1), the relation holds

$$\{n \in \mathbb{N} : r_n > 0\} \subseteq \{n \in \mathbb{N} : r_n = |r_n|\}.$$

Therefore, by Definition (6), we have

$$\{n \in \mathbb{N} : r_n = |r_n|\} \in \mathcal{U},$$

which means that $[(r_n)] = [(|r_n|)]$. Finally, we obtain $\big|[(r_n)]\big| = [(r_n)] = [(|r_n|)]$.

Secondly, if $[(r_n)] = 0$, then $\{n \in \mathbb{N} : r_n = 0\} \in \mathcal{U}$, and consequently, $\{n \in \mathbb{N} : |r_n| = 0\} \in \mathcal{U}$. Finally, we obtain $\big|[(r_n)]\big| = 0^* = [(|r_n|)]$.

Thirdly, if $[(r_n)] < 0^*$, then $-[(r_n)] = [(-r_n)] > 0^*$. Based on the first case, we have $\big|[(-r_n)]\big| = [(|-r_n|)]$. Finally, we obtain

$$\big|[(r_n)]\big| = \big|[(-r_n)]\big| = [(|-r_n|)] = [(|r_n|)].$$

Remark 4
It is easily shown that the following rules hold

$$(r \pm s)^* = r^* \pm s^*,\ (r \cdot s)^* = r^* \cdot s^*,\ (r/s)^* = r^*/s^*,\ |r|^* = |r^*|,\ (s^*)^{-1} = (s^{-1})^*,$$

for every $r, s \in \mathbb{R}$, with $s \neq 0$.

Remark 5
As we mentioned above, the field of fractions $(\mathbb{Q}, +, \cdot, 0, 1, <)$ can be embedded into every ordered field, therefore symbols such as $\frac{n}{m}$ make sense in the field of

hyperrelas. This is why, from now on we use $q$, in particular 0, 1, $n$, for hyperreal number $q^*$, where $q \in \mathbb{Q}$.

### 3.1. Infinitesimals, infinitely large, limited hyperreals

Definition 7

Sets of infinitesimals $\Omega$, limited hyperreals $\mathbb{L}$, and infinitely large hyperreals $\Psi$ are subsets of $\mathbb{R}^*$ given by

$$\begin{aligned} x \in \Omega &\Leftrightarrow (\forall n \in \mathbb{N})(|x| < 1/n), \\ x \in \mathbb{L} &\Leftrightarrow (\exists n \in \mathbb{N})(|x| < n), \\ x \in \Psi &\Leftrightarrow (\forall n \in \mathbb{N})(|x| > n). \end{aligned}$$

Since real numbers obey the Archimedean axiom, the set of infinitesimals can be also given by

$$x \in \Omega \Leftrightarrow (\forall \theta \in \mathbb{R}_+)(|x| < \theta^*). \tag{10}$$

To summarize our previous arguments, we present below a figure representing the ultraproduct construction as well as the sets $\mathbb{R}^*$, $\Omega$, $\mathbb{L}$, and $\Psi$.

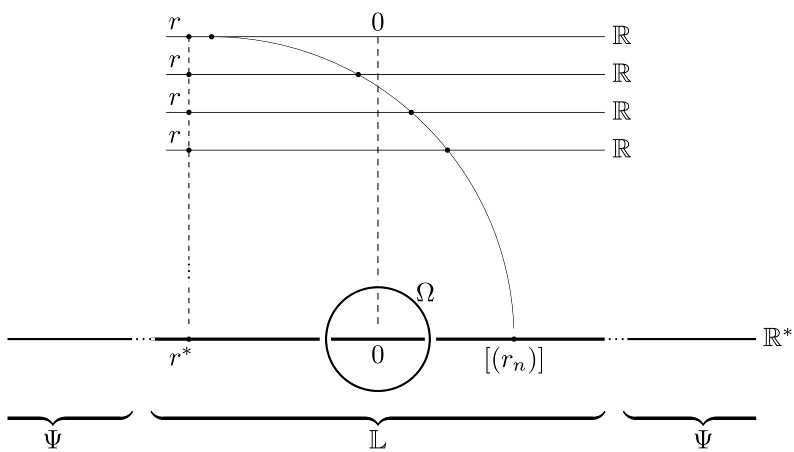


**Fig. 1.**

It is instructive to offer some simple examples of infinitesimals and infinitely large numbers. We show them in the form of the following lemma.

Lemma 2

*Let $(r_n) \subset \mathbb{R}$ be a sequence of reals. If $\lim\limits_{n\to\infty} r_n = 0$, then $[(r_n)] \in \Omega$. If $\lim\limits_{n\to\infty} r_n = \infty$, then $[(r_n)] \in \Psi$.*

*Proof.* We prove the first part of the above thesis. Let $\theta \in \mathbb{R}_+$. Since $\lim\limits_{n\to\infty} r_n = 0$, there exits $k \in \mathbb{N}$ such that $|r_n| < \theta$, for every $n > k$. Therefore,

$$\{k+1, k+2, ...\} \subseteq \{n \in \mathbb{N} : |r_n| < \theta\}.$$

It means that $\{n \in \mathbb{N} : |r_n| < \theta\} \in \mathcal{U}$ and, by (5), we obtain $[(r_n)] < \theta^*$.

The idea of the Lemma 2 is presented by the following figures.

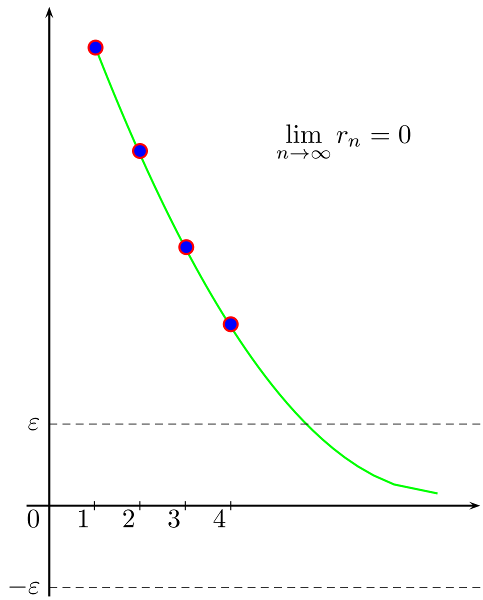


**Fig. 2.**

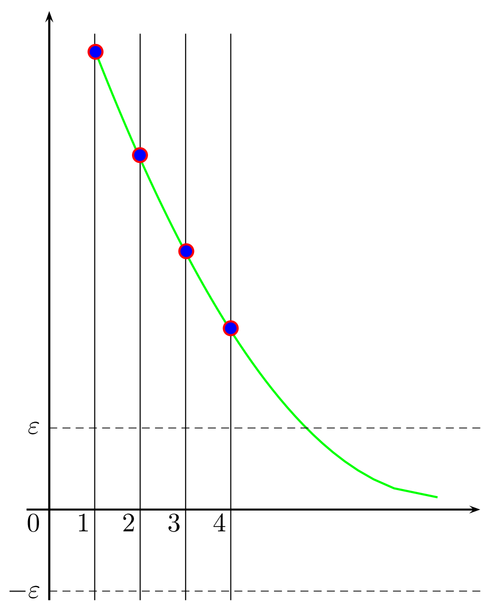


**Fig. 3.**

In Figure 2, we find the standard representation of the sequence of reals $(r_n)$ converging to the limit 0. Figure 3 below contains the previous representation with vertical axis $\{1\} \times \mathbb{R}$, $\{2\} \times \mathbb{R}$ etc. included.

Figure 4 depicts the previous diagram rotated by $\pi/2$ about the origin of the line $\mathbb{R}$, that is, the point $(0,0)$, with a part of the line of hyperreals $\mathbb{R}^*$ added. On the line $\mathbb{R}^*$, the hyperreal number $[(r_n)]$ is represented by the red dot and it lies infinitely close to the hyperreal 0.

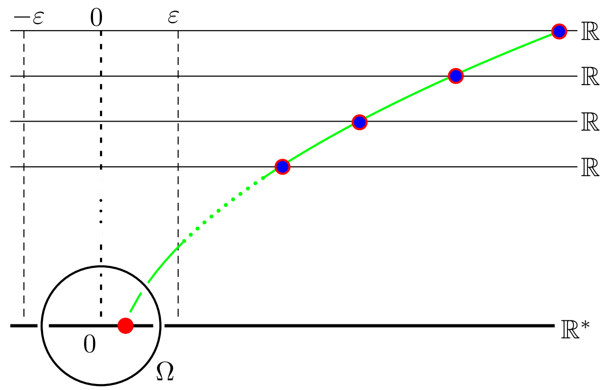


**Fig. 4.**

### 3.2. Algebra of infinitesimals

It easily follows from the very definitions that the structure $(\Omega, +, 0)$ is a group and the structure $(\mathbb{L}, +, \cdot, 0, 1)$ is a ring. Indeed, if $\varepsilon, \delta \in \Omega$ and $n \in \mathbb{N}$, then $|\varepsilon| < \frac{1}{2n}$, $|\delta| < \frac{1}{2n}$. Since $|\varepsilon + \delta| < |\varepsilon| + |\delta|$, we obtain $|\varepsilon + \delta| < \frac{1}{n}$. Similarly we show the other cases.

As a consequence of the interplay between the quantifiers "for al" and "exists" occurring in the definitions of sets $\Omega$ and $\mathbb{L}$, we obtain

$$(\forall x \in \Omega)(\forall y \in \mathbb{L})(x \cdot y \in \Omega).$$

Indeed, since $|y| < k$, for some $k \in \mathbb{N}$, and for every $n \in \mathbb{N}$ obtains $|x| < \frac{1}{nk}$, the following inequality $|x \cdot y| < \frac{1}{kn}k$ holds. Thus, $|x \cdot y| < \frac{1}{n}$, for every $n \in \mathbb{N}$.

Now, we show something more, namely

Lemma 3
*The group of infinitesimals is the maximal ideal of limited hyperreals.*

*Proof.* Seeking a contradiction, suppose that $G$ is an ideal of $\mathbb{L}$ such that $\Omega \subsetneq G$ and $G \subsetneq \mathbb{L}$. Take $x \in G \setminus \Omega$. Since $x$ is limited, the inequality $|x| < n$ holds for some $n \in \mathbb{N}$; since it is not infinitesimal, the inequality $|x| > 1/k$ holds for some $k \in \mathbb{N}$. Thus

$$1/k < |x| < m.$$

Hence, via the rules of an ordered field, we obtain

$$1/m < |x^{-1}| < k,$$

which means that $x^{-1}$ is a limited hyperreal. Since $x$ belongs to the ideal $G$ of the ring of limited hyperreals, the element $1 = x \cdot x^{-1}$ also belongs to $G$, which contradicts the fact $G \subsetneq \mathbb{L}$.

In fact, the argument given above shows that

$$x \in \mathbb{L} \setminus \Omega \Leftrightarrow x^{-1} \in \mathbb{L} \setminus \Omega. \tag{11}$$

On the set $\mathbb{R}^*$ we define a binary relation *is infinitely close* by putting

$$x \approx y \Leftrightarrow x - y \in \Omega. \tag{12}$$

When $x \approx y$ we say that $x$ is *infinitely close* to $y$. Since $(\Omega, +, 0)$ is a group, it follows that $\approx$ is an equivalence relation.

Note, that two standard real numbers $r, s$ do not lie infinitely close to each other. Indeed, it follows from the Archimedean axiom that the real number $|r - s|$ is greater than $1/k$, for some $k \in \mathbb{N}$. Therefore, the number $|r - s|^*$ is not infinitesimal, and neither $r^* - s^*$ nor $s^* - r^*$ belong to $\Omega$.

Theorem 4
*Every limited hyperreal is infinitely close to exactly one real number*

$$(\forall a \in \mathbb{L})(\exists! r \in \mathbb{R})(a \approx r^*).$$

*Proof.* The limited number $a \in \mathbb{R}^*$ determines a Dedekind cut $(L, U)$ of $(\mathbb{R}, <)$, where

$$L = \{x \in \mathbb{R} : x^* \leqslant a\}, \quad U = \{x \in \mathbb{R} : x^* > a\}.$$

By (C1), the cut $(L, U)$ determines the real number $r$ such that

$$(\forall y \in L)(\forall z \in U)(x \leqslant r \leqslant y).$$

By (A5), we find a sequence $(r_n) \subset \mathbb{R}$ such that $r_{2k-1} \in L$, $r_{2k} \in U$ and $r_{2k} - r_{2k-1} < 1/k$. Hence,

$$r^*, a \in \bigcap_{k=1} [r_{2k-1}, r_{2k}],$$

which gives $r^* \approx a$.

Since two standard reals do not lie infinitely close to each other, there exists exactly one real number infinitely close to $a$.

The unique standard real number that lies infinitely close to the limited number $a$ is called the standard part, or the shadow, of $a$ and is denoted by $a^o$. Thus, $(a^o)^* \approx a$, or $a = (a^o)^* + \varepsilon$, for some $\varepsilon \in \Omega$. Also note, that if $a \approx b$, then, by the uniqueness of the standard part, we obtain $a^o = b^o$.

If $\varepsilon$ is infinitesimal, then $\varepsilon^o = 0$.

Remark 6
By Theorem 4, one can show that the quotient ring $\mathbb{L}/\Omega$ is isomorphic to the field of real numbers. As a result, the set of limited hyperreals $\mathbb{L}$ can be represented as the sum of disjoint sets $r^* + \Omega$, the so-called monads, for $r \in \mathbb{R}$.

## 4. Calculus without limits

The set of infinitely large hypernaturals is defined by

$$\mathbb{N}_\infty = \mathbb{N}^* \setminus \mathbb{N}. \tag{13}$$

Thus, the set of hypernaturals is represented as the sum of disjoined sets, namely

$$\mathbb{N}^* = \mathbb{N}_\infty \cup \mathbb{N}.$$

Lemma 4
*Let $(k_j)$ be a sequence of natural numbers and $K = [(k_j)]$, then*

$$K \in \mathbb{N}_\infty \Leftrightarrow (\forall k \in \mathbb{N})(K > k).$$

*Proof.* Suppose that the right side of the above equivalence does not hold, that is, $K \leqslant k$, for some $k \in \mathbb{N}$. Then, by (5), we obtain $\{j \in \mathbb{N} : k_j \leqslant k\} \in \mathcal{U}$. Therefore, since

$$\{j \in \mathbb{N} : k_j = 1\} \cup ... \cup \{j \in \mathbb{N} : k_j = k\} = \{j \in \mathbb{N} : k_j \leqslant k\},$$

and sets

$$\{j \in \mathbb{N} : k_j = 1\}, ..., \{j \in \mathbb{N} : k_j = k\}$$

are pairwise disjoined, by Theorem 2, exactly one of them belongs to $\mathcal{U}$. Hence,

$$(\exists! i)(1 \leqslant i \leqslant k \wedge \{j \in \mathbb{N} : k_j = i\} \in \mathcal{U}).$$

Thus, we have $K = i$, for some $i \in \mathbb{N}$. As a result, we obtain $K \notin \mathbb{N}_\infty$.

Conversely, let $K = [(k_j)]$, for $(k_j) \subset \mathbb{N}$. Suppose that $K \notin \mathbb{N}_\infty$. Therefore, since $\mathbb{N}_\infty = \mathbb{N}^* \setminus \mathbb{N}$, we have $K = n$, for some $n \in \mathbb{N}$. This means that it is not the case that $K > k$, for every $k \in \mathbb{N}$.

Lemma 5
*If $(k_j)$ is an increasing sequence of natural numbers, then $[(k_j)] \in \mathbb{N}_\infty$.*

*Proof.* We show that $K > k$, for every $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Since the sequence $(k_j)$ is increasing, there exists $j_0$ such that $k_j > k$, for every $j > j_0$. Now, we have

$$\{j \in \mathbb{N} : j > j_0\} \subseteq \{j \in \mathbb{N} : k_j > k\}.$$

Since $\{j \in \mathbb{N} : j > j_0\}$ belongs to $\mathcal{U}$, the set $\{j \in \mathbb{N} : k_j > k\}$ also belongs to $\mathcal{U}$. By (5), this means that $[(k_j)] > k$.

Definition 8
Let $(r_n) \subset \mathbb{R}$ be a sequence of reals. The extension of $(r_n)$ to a hypersequence $(r_K) \subset \mathbb{R}^*$ is given by

$$\mathbb{N}^* \ni K \mapsto r_K \in \mathbb{R}^*,$$

where $K = [(k_j)] = [(k_1, k_2, ..., k_j, ...)]$ and $r_K = [(r_{k_1}, r_{k_2}, ..., r_{k_j}, ...)]$.

Note that, if $K = n$, for $n \in \mathbb{N}$, than $r_K = [(r_n, r_n, ...)] = r_n^*$. This means that the hypersequence $\{r_K : K \in \mathbb{N}^*\}$ includes all the terms of the sequence $\{r_n : n \in \mathbb{N}\}$, given that they are taken in the form $r_n^*$.

Definition 9
Let $f : \mathbb{R} \mapsto \mathbb{R}$. The extension of $f$ to a function $f^* : \mathbb{R}^* \mapsto \mathbb{R}^*$ is given by

$$f^*([(r_n)]) = [(f(r_n))].$$

Note that for every $r \in \mathbb{R}$ equalities hold $f^*(r^*) = [(f(r), f(r), ...)] = (f(r))^*$.
We can now present our basic theorem.

## 4.1. Basics of the calculus

Theorem 5
*If $(r_n)$ is a sequence of real numbers and $g \in \mathbb{R}$, then*

$$\lim_{n\to\infty} r_n = g \Leftrightarrow (\forall K \in \mathbb{N}_\infty)(r_K \approx g^*).$$

*Proof.* Let $\lim\limits_{n\to\infty} r_n = g$ and $K = [(k_j)]$ be an infinite hypernatural number. In order to show that $r_K \approx g^*$, or that $|r_K - g^*| \in \Omega$, by (10), we need to show that

$$(\forall \theta \in \mathbb{R}_+)(|r_K - g^*| < \theta^*).$$

Since $r_K = [(r_{k_1}, r_{k_2}, ..., r_{k_j}, ...)]$, by (5) and (6), it is equivalent to

$$(\forall \theta \in \mathbb{R}_+)(\{j \in \mathbb{N} : |r_{k_j} - g| < \theta\} \in \mathcal{U}).$$

Let $\theta \in \mathbb{R}_+$. Since $\lim\limits_{n\to\infty} r_n = g$, there exists $k$, such that $|r_n - g| < \theta$, for every $n > k$. The following relations are obvious

$$\begin{aligned}
\mathbb{N} \setminus \{1, ..., k\} &= \{j \in \mathbb{N} : j > k\}, \\
\mathbb{N} \setminus \{1, ..., k\} &\in \mathcal{U}, \\
\{j \in \mathbb{N} : j > k\} &\in \mathcal{U}, \\
\{j \in \mathbb{N} : j > k\} &\subseteq \{j \in \mathbb{N} : |r_j - g| < \theta\}, \\
\{j \in \mathbb{N} : |r_j - g| < \theta\} &\in \mathcal{U}.
\end{aligned}$$

Since $K \in \mathbb{N}_\infty$, by Lemma 4, the inequality $K > k$ holds. This means that $\{j \in \mathbb{N} : k_j > k\} \in \mathcal{U}$. Therefore, since

$$\begin{aligned}
&\{j \in \mathbb{N} : k_j > k\} \cap \{j \in \mathbb{N} : |r_j - g| < \theta\} \in \mathcal{U}, \\
&\{j \in \mathbb{N} : k_j > k\} \cap \{j \in \mathbb{N} : |r_j - g| < \theta\} \subseteq \{j \in \mathbb{N} : |r_{k_j} - g| < \theta\},
\end{aligned}$$

we finally obtain $\{j \in \mathbb{N} : |r_{k_j} - g| < \theta\} \in \mathcal{U}$, that is $r_K \approx g^*$.

For the second part of the proof we apply the *reductio ad absurdum* argument. Seeking a contradiction, suppose that $r_K \approx g^*$, for every $K \in \mathbb{N}_\infty$, and $\lim\limits_{n\to\infty} r_n \neq g$.

Since $g$ is not a limit of the sequence $(r_n)$, there exists $\theta \in \mathbb{R}_+$ such that

$$(\forall k \in \mathbb{N})(\exists n \in \mathbb{N})(n > k \wedge |r_n - g| \geqslant \theta).$$

Now, we define an increasing sequence of indexes $(k_j)$ such that the inequality $|r_{k_j} - g| \geqslant \theta$ holds, for every $j \in \mathbb{N}$. Set $K = [(k_j)]$. By Lemma 5, $K \in \mathbb{N}_\infty$. Moreover, we obviously have

$$\{j \in \mathbb{N} : |r_{k_j} - g| \geqslant \theta\} = \mathbb{N},$$
$$\{j \in \mathbb{N} : |r_{k_j} - g| \geqslant \theta\} \in \mathcal{U}.$$

Therefore, $|r_K - g^*| \geqslant \theta^*$. This means that there exists $K \in \mathbb{N}_\infty$ such that $|r_K - g^*| \geqslant \theta^*$. This contradicts the fact that $|r_K - g^*| < \theta^*$, for every $K \in \mathbb{N}_\infty$.

The idea of the first part of the Theorem 5 is presented by the Figures 5 to 8 below.

If $\lim\limits_{n\to\infty} r_n = g$, then every subsequence of $(r_n)$ also converges to the limit $g$. In Figure 5 the convergence of a subsequence $(r_{k_j})$ to the limit $g$ is presented. Moreover, on the horizontal line $\mathbb{R}\times\{0\}$ the sequence of indexes $(k_j) = (k_1, k_2, ...)$ is indicated. Figure 6 contains the previous representation with vertical axis $\{k_1\}\times\mathbb{R}$, $\{k_2\} \times \mathbb{R}$ etc. included.

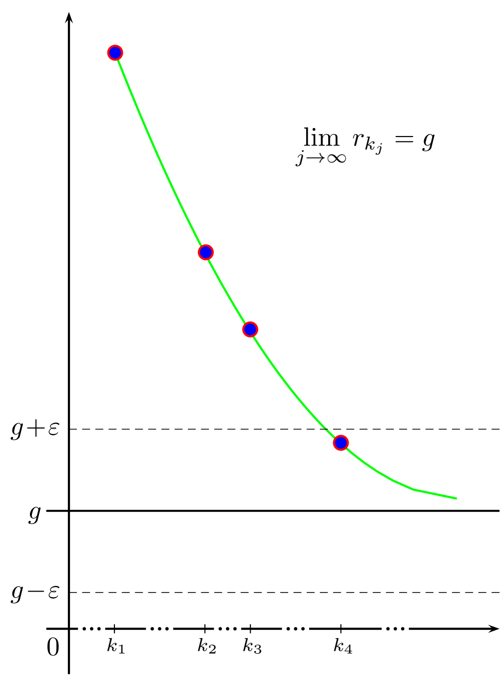


**Fig. 5.**

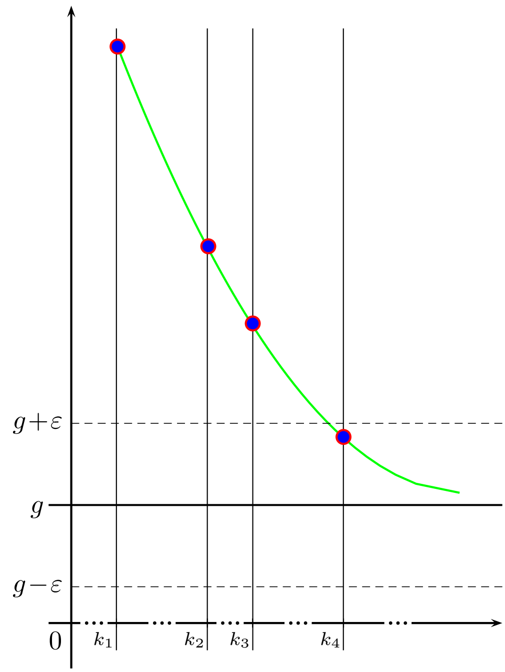


**Fig. 6.**

Figure 7 depicts the previous diagram rotated by $\pi/2$ about the origin of the line $\mathbb{R}$, that is, the point $(0,0)$. Note that there are no terms of the subsequence $(r_{k_j})$ on the axes $\{k\} \times \mathbb{R}$, where $k_j < k < k_{j+1}$.

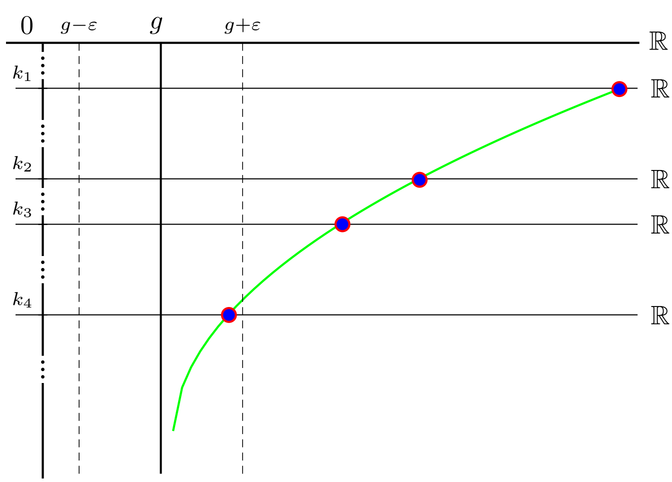


**Fig. 7.**

In Figure 8 the previous diagram is *squeezed into* the ultraproduct: the term $r_{k_1}$ is shifted from the axis $\{k_1\} \times \mathbb{R}$ to the axis $\{1\} \times \mathbb{R}$, the term $r_{k_2}$ is shifted from the axis $\{k_2\} \times \mathbb{R}$ to the axis $\{2\} \times \mathbb{R}$ etc. As a result, $(r_{k_1}, r_{k_2}, ...) \in \mathbb{R}^{\mathbb{N}}$. The hyperreal number $r_K = [(r_{k_j})]$ is represented by the red dot on the line $\mathbb{R}^*$.

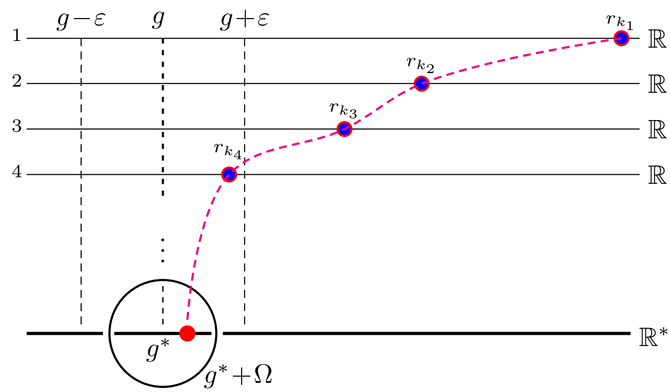


**Fig. 8.**

Similarly, one can prove the following theorems:

Theorem 6
*If $f : \mathbb{R} \mapsto \mathbb{R}$ and $r \in \mathbb{R}$, then the following equivalence holds:*

$$f \text{ is continuous at point } r \Leftrightarrow (\forall x \in \mathbb{R}^*)(x \approx r^* \Rightarrow f^*(x) \approx f^*(r^*)),$$
$$f \text{ is uniformly continuous on } \mathbb{R} \Leftrightarrow (\forall x, y \in \mathbb{R}^*)(x \approx y \Rightarrow f^*(x) \approx f^*(y)).$$

Theorem 7
*If $f : \mathbb{R} \mapsto \mathbb{R}$ and $r \in \mathbb{R}$, then the real number $L \in \mathbb{R}$ is a derivative of $f$ at $r$ if and only if for every nonzero infinitesimal $\varepsilon$ the relation holds*

$$\frac{f^*(r^* + \varepsilon) - f^*(r^*)}{\varepsilon} \approx L^*.$$

In the theses of Theorems 5 to 7, the basic notions of calculus, that is, the limit of sequence, the continuity of function, and the derivative of real function find their equivalent formulations in a language that applies notions of infinitesimals, infinitely large numbers and the relation *is infinitely close*.

## 4.2. Algebra of limits

In this section we apply algebra of infinitesimals developed in § 3.2 to the algebra of limits. Proving Theorem 8 below, we proceed as follows. By Theorem 5, the assumption $\lim_{n\to\infty} r_n = g$, $\lim_{n\to\infty} s_n = h$ is equivalent to the claim that $r_K \approx g^*$, $s_K \approx h^*$ for every $K \in \mathbb{N}_\infty$. By Lemma 3, we show that algebraic operations carried out on hyperreals $r_K - g^*$, $s_K - h^*$, $r_K, s_K, g^*, h^*$ give results such as $r_K + s_K \approx (g+h)^*$, $r_K s_K \approx (gh)^*$, $r_K/s_K \approx (g/h)^*$. Then, reapplying Theorem 5, we get $\lim_{n\to\infty}(r_n + s_n) = g + h$ etc.

Theorem 8
*If* $\lim\limits_{n\to\infty} r_n = g$ *and* $\lim\limits_{n\to\infty} s_n = h$*, then*

1. $\lim\limits_{n\to\infty} (r_n + s_n) = g + h,$
2. $\lim\limits_{n\to\infty} (r_n \cdot s_n) = gh,$
3. $\lim\limits_{n\to\infty} r_n/s_n = g/h$*, where* $h \neq 0$ *and* $s_n \neq 0$ *for every* $n \in \mathbb{N}$*.*

*Proof.* Since the sequence $(r_n)$ converges to $g$, it is a limited sequence, that is, $|r_n| < M$, for every $n \in \mathbb{N}$ and some $M \in \mathbb{R}$. Thus for every $K = [(k_j)]$ the equality obtains $\{j \in \mathbb{N} : |r_{k_j}| < M\} = \mathbb{N}$. This means that every hyperreal number $r_K$ is limited. Similarly, $s_K \in \mathbb{L}$, for every $K \in \mathbb{N}^*$.

Let $K = [(k_j)] \in \mathbb{N}_\infty$. If $\lim\limits_{n\to\infty} r_n = g$ and $\lim\limits_{n\to\infty} s_n = h$, then, by Theorem 4, we have $r_K \approx g^*$ and $s_K^* \approx h^*$, or equivalently,

$$r_K - g^* \in \Omega, \quad s_K^* - h^* \in \Omega.$$

1. Since infinitesimals form a group, we obtain that the sum of $r_K - g^*$ and $s_K - h^*$ is infinitesimal. This means that $r_K + s_K \approx (g+h)^*$. By Theorem 4, it gives $\lim\limits_{n\to\infty} (r_n + s_n) = g + h$.

2. Since $r_K - g^*$ is infinitesimals and $s_K$ is limited, by Lemma 3, we obtain

$$(r_K - g^*)s_K \in \Omega.$$

Similarly, since $s_K - h^*$ is infinitesimals and $g^*$ is limited, by Lemma 3, we obtain,

$$(s_K - h^*)g^* \in \Omega.$$

Adding these two infinitesimals, we have $r_K s_K - (gh)^* \in \Omega$, or equivalently,

$$r_K s_K \approx (gh)^*.$$

By Theorem 4, it gives $\lim\limits_{n\to\infty} r_n s_n = gh$.

3. Let $h \neq 0$. Since $\lim\limits_{n\to\infty} s_n = h$ and $s_n \neq 0$ for every $n \in \mathbb{N}$, then for some $d, D > 0$ equality obtains $\{n \in \mathbb{N} : D > |s_n| > d\} = \mathbb{N}$. By the rules of ordered field it follows that $\{n \in \mathbb{N} : 1/d > 1/|s_n| > 1/D\} = \mathbb{N}$. Thus $1/s_K \in \mathbb{L}$, for every $K \in \mathbb{N}^*$. Since $h^* \in \mathbb{L} \setminus \Omega$, by (11) we have $1/h^* \in \mathbb{L}$. As a result, $(1/s_k)(1/h^*) \in \mathbb{L}$.

Now,

$$\frac{1}{s_K} - \frac{1}{h^*} = \frac{h^* - s_K}{s_K s^*}.$$

Since $h^* - s_K$ is infinitesimal and $1/(s_K s^*)$ is limited, we obtain $1/s_K - 1/h^* \in \Omega$, or equivalently, $1/s_K \approx 1/h^*$. By Theorem 4, it gives that $\lim\limits_{n\to\infty} 1/s_n = 1/h$. To end the proof is enough to apply case 2 to sequences $(r_n)$ and $(1/s_n)$.

## References

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*Instytut Matematyki*
*Uniwersytet Pedagogiczny*
*ul. Podchorążych 2*
*PL-30-084 Kraków*
*e-mail:* pb@up.krakow.pl
*e-mail:* jmajor@up.krakow.pl