

A PURELY ALGEBRAIC PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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ABSTRACT. Proofs of the fundamental theorem of algebra can be divided up into three groups according to the techniques involved: proofs that rely on real or complex analysis, algebraic proofs, and topological proofs. Algebraic proofs make use of the fact that odd-degree real polynomials have real roots. This assumption, however, requires analytic methods, namely, the intermediate value theorem for real continuous functions. In this paper, we develop the idea of algebraic proof further towards a purely algebraic proof of the intermediate value theorem for real polynomials. In our proof, we neither use the notion of continuous function nor refer to any theorem of real and complex analysis. Instead, we apply techniques of modern algebra: we extend the field of real numbers to the non-Archimedean field of hyperreals via an ultraproduct construction and explore some relationships between the subring of limited hyperreals, its maximal ideal of infinitesimals, and real numbers.

1. Introduction. In 1799, Gauss gave the first widely accepted proof of the fundamental theorem of algebra, FTA for short: Every nonconstant complex polynomial has a complex root. Since then, many new proofs have appeared, including new insights, as well as a diversity of tricks, techniques and general methods. Nearly a hundred proofs of FTA were published up to 1907 (see [20, p. 98]). Another hundred proofs were released in the period 1933 to 2009 (see [23]). This unusual number of proofs compares with the multitude of proofs of the Pythagorean theorem (see [19]).

Fine and Rosenberg [11] take a more general, qualitative perspective and present six exemplary proofs of FTA classified according to the techniques involved. They pair these proofs in accordance with the basic areas of mathematics and present these pairs as models of analytic, algebraic and topological proofs. The first two proofs require real and complex analysis, the third and fourth ones apply algebraic methods: splitting fields and the fundamental theorem of symmetric polynomials, or the Galois theory and the Sylow theorem. The fifth proof involves the notion of the winding number of a closed, continuously differentiable

curve $f : \mathbb{R} \rightarrow \mathbb{C}$ around 0. The sixth one relies on the algebraic topology and applies the Brouwer fixed point theorem.¹

After the presentation of analytic and algebraic proofs, Fine and Rosenberg observe:

“We have now seen four different proofs of the Fundamental Theorem of Algebra. The first two were purely analysis, while the second pair involved a wide range collection of algebraic ideas. However, we should realize that even in these proofs we did not totally leave analysis. Each of these proofs used the fact the odd-degree real polynomials have real roots. This fact is a consequence of the intermediate value theorem, which depends on continuity. Continuity is a topological property and we now proceed to our final pair of proofs, which involve topology.” [11, p. 134]

All these proofs mentioned above involve, however, two kinds of continuity: the continuity of total order and the continuity of function. The first one is the characteristic feature of real numbers, and since we deal with real polynomials, we cannot ignore the continuity of the reals. The second one refers to a function – one can call the continuity of this kind a *topological property*. In our proof of FTA, the continuity of a function is omitted. In the next section, we develop this distinction further.

2. Two kinds of continuity. The continuity of the field of real numbers can be formulated in many equivalent ways. In this paper, we apply what we believe to be the simplest development – the one introduced by Richard Dedekind in his 1872 [10]. To this end, we need a notion of Dedekind cut.

Definition. A pair of sets (L, U) is a Dedekind cut of a totally ordered set $(X, <)$ if (1) $L, U \neq \emptyset$, (2) $L \cup U = X$, (3) $(\forall y \in L)(\forall z \in U)(y < z)$. A cut (L, U) is called a gap if there exists neither a maximum in L nor a minimum in U . A cut (L, U) is called a jump if there exists both a maximum in L and a minimum in U .

Now, depending on whether we consider the real line $(\mathbb{R}, <)$ or the field of reals $(\mathbb{R}, +, \cdot, 0, 1, <)$, we get different continuity axioms. The categorical characteristics of a continuously ordered set $(X, <)$, due to George Cantor’s [6], consists of three conditions: (1) the order is dense, (2) no Dedekind cut of $(X, <)$ is a gap, (3) the order is separable, i.e.,

¹Some of the recent results use linear algebra [9] and nonstandard analysis [16]. The first one relies on the intermediate value theorem for real continuous functions, the second one exploits the Brouwer fixed point theorem.

there exists such a countable subset $Z \subset X$ that is dense in X ,

$$(\forall x, y \in X)(\exists z \in Z)(x < y \Rightarrow x < z < y).^2$$

In other words, any ordered set $(X, <)$ that satisfies the three above conditions is isomorphic to the line of real numbers $(\mathbb{R}, <)$.

The continuity axiom for ordered fields is significantly simpler, for it consists of the requirement (2) alone. This is because, the density of the field order follows from ordered field axioms, while the continuity axiom itself implies that the order is separable (see below).

From now on, \mathcal{F} denotes a totally ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$, that is a commutative field with a total order that is compatible with addition and multiplication,

$$\begin{aligned} &(\forall x, y \in \mathbb{F})(x < y \Rightarrow x + z < y + z), \\ &(\forall x, y, z \in \mathbb{F})(x < y, 0 < z \Rightarrow x \cdot z < y \cdot z). \end{aligned}$$

Definition. The field of real numbers is an ordered field \mathcal{F} , in which every Dedekind cut (L, U) of $(\mathbb{F}, <)$ satisfies the condition

$$(C1) \quad (\exists x \in \mathbb{F})(\forall y \in L)(\forall z \in U)(y \leq x \leq z).$$

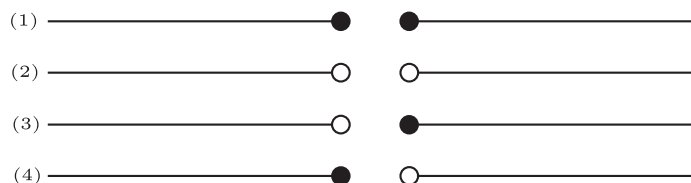
The categoricity theorem states that any two ordered fields satisfying axiom (C1) are isomorphic ([7, p. 105]). In other words, any ordered field satisfying (C1) is isomorphic to the field of real numbers $(\mathbb{R}, +, \cdot, 0, 1, <)$.

The well-known constructions of the reals, e.g., the one that identifies real numbers with cuts of the line of rational numbers $(\mathbb{Q}, <)$ due to Dedekind, show that there exists at least one field of real numbers. On the other hand, the categoricity theorem implies that there exists at most one, up to isomorphism, field of real numbers.

The order of an ordered field is dense: it follows from the simple observation that if $x < y$, then $(x + y)/2$ lies between x and y . On the other hand, the density of a field order is equivalent to the claim that no cut of $(\mathbb{F}, <)$ is a jump. As a result, (C1) is reduced to the fact that no Dedekind cut of $(\mathbb{F}, <)$ is a gap.³ With this knowledge we can easily visualize the continuity of the real line.

²Next to [6], § 11. *Der Ordnungstypus θ des Linearkontinuums X* , see also [15], ch. VI, § 3. *Order types ω, η , and λ* .

³It follows from the axiom (C1) that the set of fractions, \mathbb{Q} , is dense in $(\mathbb{R}, <)$; in fact, it is an equivalent form of the Archimedean axiom (see § 4 below). Thus, for ordered fields, (C1) suffice to characterize the continuity of the order.



Dedekind cuts

On the above diagram a black dot corresponds to the maximal element of the class L , or the minimal element of the class U . If there are, accordingly, no maximum or minimum, then we put a white dot. In this way the diagram *Dedekind cuts* represents the only possible kinds of cuts of the line $(\mathbb{F}, <)$. Cut (1) is a jump, cut (2) is a gap. Axiom (C1) states that cuts (3) and (4) are the only possible cuts on the line of real numbers $(\mathbb{R}, <)$.

(C1) is called the continuity axiom. In fact, it is one of the many equivalent formulations of the continuity of the reals. Here are some other popular versions:

(C2) If $A \subset \mathbb{F}$ is a nonempty set which is bounded above, then there exists $a \in \mathbb{F}$ such that $a = \sup A$.

(C3) Field \mathcal{F} is Archimedean and every Cauchy (fundamental) sequence $(a_n) \subset \mathbb{F}$ has a limit in \mathbb{F} .

Cohn and Ehrlich [7, p. 95–96], present other standard formulations of the continuity axiom which appeared at the turn of 19th and 20th century. Due to their existential character (“there exists $x \in \mathbb{F}$ such that”), they all make it possible to determine specific real numbers – each version with a different technique, e.g., via Dedekind cuts, bounded sets, or fundamental sequences. In a sense, each form of the continuity axiom corresponds to a technique. In the next section we present a new version of the continuity axiom that reflects the technique we adopt in our proof of FTA.

2.1. Now we turn to the continuity of function.

Definition. A function $f : \mathbb{F} \mapsto \mathbb{F}$ is continuous at a point $a \in \mathbb{F}$ if the following condition holds

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{F})(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon).$$

To elaborate, first, observe that the definition of a continuous function can be formulated in any ordered field. Similarly, we can develop the theory of limits of sequences in any order field (see [7], chap. 3). Moreover, in any ordered field we can also formulate the intermediate value theorem (IVT for short).

Secondly, the standard proof of IVT for a real continuous function proceeds as follows: Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function with $f(0) < 0$ and $f(1) > 0$. Putting $A = \{x \in [0, 1] : f(x) < 0\}$, by (C2), we can take $a = \sup A$. The continuity of f implies $f(a) = 0$. The final step in this argument is based on the rule stating that a real continuous function preserves the sign:

$$(SR) \quad f(c) \neq 0 \Rightarrow (\exists \delta > 0)(\forall x \in \mathbb{R})(|c - x| < \delta \Rightarrow f(c) \cdot f(x) > 0).$$

Thus, if $f(a) < 0$, then, by (SR), for some $\delta > 0$ and every x such that $0 < x - a < \delta$ holds $f(x) < 0$, contrary to the assumption $a = \sup A$; the same argument applies to the case $f(a) > 0$. Hence, via the trichotomy law of the total order, $f(a)$ equals 0.

This proof clearly manifests the combination of the continuity of the reals and the continuity of a function: we apply the (SR) rule to the point defined by (C2).⁴

Thirdly, from the logical point of view, the difference between the continuity of an ordered field and the continuity of function is not an absolute one. Axiom (C1) turns out to be equivalent to many statements of real analysis which are typically presented as theorems and involve the notion of a function. Teismann [22] proves that IVT, as well as the mean value theorem, is equivalent to (C1).⁵ Riemenschneider [21] lists 37 versions of the continuity axiom, mostly theorems of one-variable real analysis.

3. Real closed fields. The theory of real closed fields, started by Emil Artin and Otto Schreier in the 1920s, provides a general framework for this paper. Our sketchy account of this theory starts with a technical notion of a formally real field.⁶ At the end of this section we show that in order to prove FTA it suffices to prove IVT for polynomials.

Definition. A field $(\mathbb{F}, +, \cdot, 0, 1)$ is formally real if the sum of squares is zero only if each summand is zero.⁷

⁴Essentially, this argument goes back as far as Bolzano [5].

⁵The proof of the first equivalence can be easily based on our diagram *Dedekind cuts*. If a field \mathcal{F} does not satisfy (C1), then there is a cut of $(\mathbb{F}, <)$ which yields a gap (L, U) , as the one marked by (2). A function given by $f(x) = 0$, for $x \in L$, and $f(x) = 1$, for $x \in U$, is continuous and does not satisfy IVT. If \mathcal{F} is the real number field, then we can adopt the proof presented above to show that a real continuous function satisfies IVT.

⁶The general reference here is [1], [8], chap. 8, [14], chap. 6; see also [24] as it applies the theory of real closed field to give a constructive proof of FTA.

⁷We assume that all fields considered are commutative.

An equivalent formulation of this definition is this: A field is formally real if -1 cannot be written as a sum of squares, that is $-1 \notin \left\{ \sum a_i^2 : a_i \in \mathbb{F} \right\}$. Thus, any ordered field is obviously formally real.

Definition. A field $(\mathbb{F}, +, \cdot, 0, 1)$ is real closed if it is formally real and every proper algebraic extension of the field is not formally real.

A real closed field is in fact an ordered field. Setting

$$x < y \Leftrightarrow y - x \in \left\{ \sum a_i^2 : a_i \in \mathbb{F} \right\}$$

we obtain a total order compatible with the addition and multiplication on \mathbb{F} .

The real number field and the real algebraic number field are both really closed, so the order of a real closed field is not necessarily continuous. Next, some theorems concerning real polynomials, like the intermediate value theorem, mean value theorem, extreme value theorem, Rolle's theorem, or theorem of Sturm about the number of zeros in an interval, also hold in a real closed field.⁸ These results suggest that the algebraic features of the real number field alone imply IVT.

3.1. The theory of real closed fields provides a criterion for a field to be algebraically closed. Artin and Schreier [1] show that: "In a real closed field, every polynomial of odd degree has at least one root." [1, p. 275]

Then, they proceed to prove the proposition that constitutes the basis for our argument, namely:

"A real closed field is not algebraically closed. On the other hand, the field obtained by adjoining i ($= \sqrt{-1}$) is algebraically closed." [1, p. 275]⁹

In fact, Artin and Schreier present two proofs which make the essence of what Fine and Rosenberg call the algebraic proofs of FTA.

The above result supports the conclusion that FTA is equivalent to the claim that real numbers form a real closed field.

3.2. The definition of a real closed field does not provide simple criteria for a field to be really closed. Yet, real closed fields can be described more explicitly. Cohn [8] shows that an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ is real closed if and only if it is closed under a square root operation and odd-degree polynomials have roots in \mathbb{F} , that is

$$(R1) \quad (\forall x > 0)(\exists y \in \mathbb{F})(y^2 = x),$$

⁸See [14], chap. VI, § 3.

⁹The converse of this theorem also holds: If \mathcal{F} is such an ordered field that the field obtained by adjoining $\sqrt{-1}$ is algebraically closed, then \mathcal{F} is real closed (see [14, p. 277]).

$$(R2) (\forall a_1 \in \mathbb{F}) \dots (\forall a_{2n} \in \mathbb{F}) (\exists x \in \mathbb{F}) (x^{2n+1} + \sum_{i=1}^{2n} a_i x^i = 0).$$

Indeed, these properties, along with ordered field axioms, constitute the set of axioms for ordered real closed fields (see [18, p. 94–95]).¹⁰ It is also easily seen that properties (R1), (R2) can be combined into this one:

“An ordered field is real closed if and only if it has the intermediate value property for polynomials.” [8, p. 315]

As a result we obtain:

Proposition 1. If an ordered field has the intermediate value property for polynomials, then the field obtained by adjoining $\sqrt{-1}$ is algebraically closed.

Thus, to prove FTA it is sufficient to prove IVT for real polynomials.

4. Archimedean fields. In the next section we will deal with a non-Archimedean field, so now we restate the definition and some basic facts concerning Archimedean fields.

Definition. A totally ordered field \mathcal{F} is Archimedean if it satisfies the condition

$$(A1) \quad (\forall x \in \mathbb{F}) (\exists n \in \mathbb{N}) (n > x).$$

Axiom (A1) is called the Archimedean axiom. Here is its equivalent version

$$(A2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Yet another equivalent formulation of the Archimedean axiom is the following one: If (L, U) is a Dedekind cut of $(\mathbb{F}, <)$, then

$$(A3) \quad (\forall n \in \mathbb{N}) (\exists x \in L) (\exists y \in U) (y - x < 1/n).$$

Given (L, U) is a Dedekind cut of $(\mathbb{F}, <)$, by (A3), we can find such a sequence $(r_n) \subset \mathbb{F}$ that satisfies conditions

$$(1) \quad r_{2k-1} \in L, \quad r_{2k} \in U, \quad r_{2k} - r_{2k-1} < \frac{1}{k},$$

$$r_1 \leq r_3 \leq \dots \leq r_{2k-1} \leq \dots \leq r_{2k} \leq \dots \leq r_4 \leq r_2.$$

We refer to this fact proving proposition 3 below.

The real number field is the *biggest* Archimedean field, that is any Archimedean field can be embedded into the field of reals; on the other hand, any extension of the reals is a non-Archimedean field (see [7]).

¹⁰Given $(\mathbb{F}, +, \cdot, 0, 1)$ is formally real, the axiom (R1) has to take the following form: $(\forall x \in \mathbb{F}) (\exists y \in \mathbb{F}) (y^2 = x \vee y^2 = -x)$.

5. Extending the real number field. In this section we introduce the construction called the ultraproduct to extend the field of reals to the field of hyperreals (nonstandard real numbers). We apply this construction directly to the field of reals to build a special kind of ultraproduct called an ultrapower. This presentation follows [8], [12].

To start with, we give the definition of an ultrafilter.

Definition. A family of sets $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ is an ultrafilter on \mathbb{N} if (1) $\emptyset \notin \mathcal{U}$, (2) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$, (3) if $A \in \mathcal{U}$ and $A \subset B$, then $B \in \mathcal{U}$, (4) for each $A \subset \mathbb{N}$, either A or its complement $\mathbb{N} \setminus A$ belongs to \mathcal{U} .

It follows from this definition that either the set of odd numbers or the set of even numbers belongs to an ultrafilter.

Now, take the family of sets with finite complements. It obviously satisfies conditions (1)–(3) listed in the definition of an ultrafilter. By Zorn's lemma, this family can be extended to an ultrafilter \mathcal{U} on \mathbb{N} (see [8, p. 29], [12, p. 20–21]).

From now on, \mathcal{U} denotes a fixed ultrafilter on \mathbb{N} containing every subset with a finite complement.¹¹

In the product $\mathbb{R}^{\mathbb{N}}$ we define a relation

$$(r_n) \equiv (s_n) \Leftrightarrow \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{U}.$$

This is easily seen to be an equivalence relation, satisfying reflexivity, symmetry and transitivity. Let \mathbb{R}^* denote the reduced product $\mathbb{R}^{\mathbb{N}}/\equiv$.

The equality relation in \mathbb{R}^* is obviously given by

$$[(r_n)] = [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{U}.$$

It follows from the notion of ultrafilter that

$$[(r_n)] \neq [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : r_n \neq s_n\} \in \mathcal{U}.$$

Algebraic operations on \mathbb{R}^* are defined pointwise, that is

$$[(r_n)] + [(s_n)] = [(r_n + s_n)], \quad [(r_n)] \cdot [(s_n)] = [(r_n \cdot s_n)].$$

And a total order on \mathbb{R}^* is given by the following definition

$$[(r_n)] < [(s_n)] \Leftrightarrow \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{U}.¹²$$

We embed the set of reals \mathbb{R} into the set of hyperreals \mathbb{R}^* by identifying the standard real number r with the hyperreal determined by

¹¹The reader can find the same reasoning in the literature encoded in one short sentence: Let \mathcal{U} be a fixed nonprincipal ultrafilter on \mathbb{N} .

¹²In the above definitions, we adopt a standard convention to use the same signs for the relations and operations on \mathbb{R} and \mathbb{R}^* ; see [12], chap. 3, § 3.6.

the constant sequence (r, r, r, \dots) . Thus, setting $r^* = [(r, r, r, \dots)]$, the embedding is given by the following map

$$\mathbb{R} \ni r \mapsto r^* \in \mathbb{R}^*.$$

To end this part of our development, we would like to ease our notation: from now on we use r for hyperreal number r^* . In fact, it is a consequence of the convention that \mathbb{R} is a *subset* of \mathbb{R}^* .

Proposition 2. $(\mathbb{R}^*, +, \cdot, 0, 1, <)$ is non-Archimedean ordered field.

PROOF. Goldblatt [12, p. 23–24] gives a straightforward proof that the hyperreals form a totally ordered field.¹³ In addition, by (A2), for any $r \in \mathbb{R}$ the inequality holds $r \neq [(1/n)]$, which means that the field of hyperreals extends the field of reals. Thus, the field of hyperreals is non-Archimedean field. \square

In the last section, we will also show that the field of hyperreals is real closed.

5.1. We define, in a standard way, subsets of \mathbb{R}^* – sets of infinitesimals, limited and infinitely large hyperreals, namely¹⁴

$$\begin{aligned} x \in \Omega &\Leftrightarrow (\forall n \in \mathbb{N})(|x| < 1/n), \\ x \in \mathbb{L} &\Leftrightarrow (\exists n \in \mathbb{N})(|x| < n), \\ x \in \Psi &\Leftrightarrow (\forall n \in \mathbb{N})(|x| > n). \end{aligned}$$

The set of positive hyperintegers (hypernaturals) \mathbb{N}^* is defined by

$$\mathbb{N}^* = \{[(r_n)] \in \mathbb{R}^* : \{n \in \mathbb{N} : r_n \in \mathbb{N}\} \in \mathcal{U}\}.$$

Roughly speaking, the set \mathbb{N}^* consists of elements $[(n_j)]$, where $(n_j) \subset \mathbb{N}$.

Next, on the set \mathbb{R}^* we define a relation x is *infinitely close to* y by putting

$$x \approx y \Leftrightarrow x - y \in \Omega.$$

This is easily seen to be an equivalence relation, satisfying reflexivity, symmetry and transitivity.

Following are some elementary facts concerning these concepts. For the sake of completeness, we present a short justification for each one, though they are almost obvious.

¹³This claim is also a straightforward consequence of the Łoś theorem [17], also known as the transfer principle; however, in this paper we avoid arguments relying on mathematical logic.

¹⁴Our definitions agree with the those given by Artin and Schreier [1]. However, we can offer some simple examples of infinitesimals and infinitely large numbers, namely $[(1/n)] \in \Omega$, $[(n)] \in \Psi$, moreover, it is easy to demonstrate that if $\lim_{n \rightarrow \infty} r_n = 0$, then $[(r_n)] \in \Omega$, and if $\lim_{n \rightarrow \infty} r_n = \infty$, then $[(r_n)] \in \Psi$.

(F1) A standard real is a limited hyperreal, $\mathbb{R} \subset \mathbb{L}$.

It is the consequence of the Archimedean axiom.

(F2) Two standard real numbers r, s do not lie infinitely close to each other.

It follows from the Archimedean axiom that the real number $|r - s|$ is greater than $1/k$ for some $k \in \mathbb{N}$. Thus this number is not infinitesimal, and neither $r - s$ nor $s - r$ belongs to Ω .

(F3) Limited numbers form an ordered ring $(\mathbb{L}, +, \cdot, 0, 1, <)$ with Ω being its maximal ideal. Particular, the following condition is satisfied

$$(\forall x \in \Omega)(\forall y \in \mathbb{L})(x \cdot y \in \Omega).$$

The first part of this claim is a consequence of the interplay between the quantifiers “for all” and “exists” occurring in the definitions of sets Ω and \mathbb{L} . To show that Ω is the maximal ideal of the ring \mathbb{L} , suppose, to obtain a contradiction, that G is an ideal of \mathbb{L} such that $\Omega \subsetneq G$ and $G \subsetneq \mathbb{L}$. Take $x \in G \setminus \Omega$. Since x is limited, for some $m \in \mathbb{N}$ holds $|x| < m$; since it is not infinitesimal, for some k holds $|x| > 1/k$. Then

$$1/k < |x| < m.$$

Hence, via the rules of an ordered field, we obtain

$$1/m < |x^{-1}| < k,$$

which means that x^{-1} is a limited hyperreal. Since x belongs to the ideal G , the element $1 = x \cdot x^{-1}$ also belongs to G , a contradiction.¹⁵

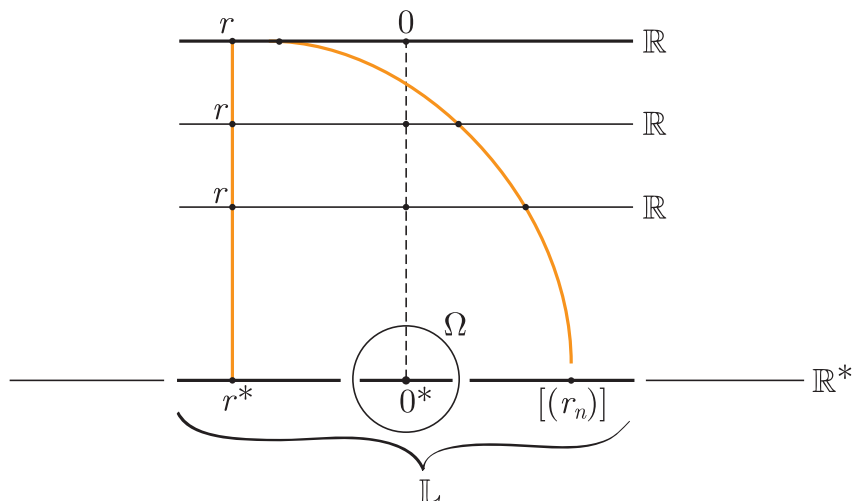
(F4) If $x \in \mathbb{R}^*$ and $x \neq 0$, then the equivalence holds

$$x \in \Omega \Leftrightarrow x^{-1} \in \Psi.$$

If $x \in \Omega \setminus \{0\}$ and $x^{-1} \notin \Psi$, then $x^{-1} \in \mathbb{L}$. By (F3), $x \cdot x^{-1} \in \Omega$, contrary to the fact that 1 is not an infinitesimal. Next, if $|x^{-1}| > n$, for every $n \in \mathbb{N}$, then $x < 1/n$, for every $n \in \mathbb{N}$. It is equivalent to the claim that if x^{-1} is infinitely large, then x is infinitely small.

To summarize this subsection, we present a diagram representing the ultrapower construction.

¹⁵By proposition 3, one can show that the quotient ring \mathbb{L}/Ω is isomorphic to the real number field. As a result, we can represent the set \mathbb{L} as the sum of disjoint sets $r + \Omega$, the so-called *monads*, for $r \in \mathbb{R}$.



5.2. We can apply the ultrapower construction to any ordered field \mathcal{F} . Thereby we obtain the set \mathbb{F}^* and its subsets Ω , \mathbb{L} , Ψ , as well as the relation *x is infinitely close to y*. With these notions we can formulate yet another version of the continuity axiom, namely

$$(C4) \quad (\forall a \in \mathbb{L})(\exists! z \in \mathbb{F})(a \approx z).$$

Proposition 3. The statements (C1) and (C4) are equivalent.

PROOF. The first part of this claim – if $\mathcal{F} = (\mathbb{R}, +, \cdot, 0, 1, <)$, then each limited number a is infinitely close to exactly one real number z – is the well-known Standard Part Principle (see [2], [12, p. 53]). In the proof that follows, we apply the Archimedean axiom in the version (A3).

The limited number $a \in \mathbb{R}^*$ determines a Dedekind cut of $(\mathbb{R}, <)$

$$L = \{x \in \mathbb{R} : x \leq a\}, \quad U = \{x \in \mathbb{R} : x > a\}.$$

By (C1), the cut (L, U) determines the real number z . By (1), we find a sequence $(r_n) \subset \mathbb{R}$ such that $r_{2k-1} \in L$, $r_{2k} \in U$ and $r_{2k} - r_{2k-1} < 1/k$. Hence,

$$z, a \in \bigcap_{k=1} [r_{2k-1}, r_{2k}],$$

which gives $z \approx a$.

Since two standard reals do not lie infinitely close to each other, only one real is infinitely close to a . This unique number is called the standard part, or the shadow, of a and is denoted by a^o . Thus $a^o \approx a$, or $a = a^o + \varepsilon$, for some $\varepsilon \in \Omega$. Also, note that if $a \approx b$, then, by the uniqueness of the standard part, we obtain $a^o = b^o$.

For the second part of the proof, we first show that (C4) implies (A2).¹⁶ Suppose, on the contrary, that (A2) does not hold. Then, for some $\varepsilon > 0$ and for every n holds $\varepsilon < 1/n$. Hence $0 \approx [(1/n)] \approx \varepsilon$, which contradicts the claim that there is only one element in \mathbb{F} infinitely close to $[(1/n)]$.

Thus we come to the main part of the proof. Let (L, U) be a Dedekind cut of $(\mathbb{F}, <)$ and $(r_n) \subset \mathbb{F}$ satisfy condition (1). Then $r_1 < [(r_n)] < r_1 + 1$. Since \mathcal{F} is Archimedean, for some $n \in \mathbb{N}$ holds $[(r_n)] < n$. As a result we obtain that $[(r_n)] \in \mathbb{L}$. By (C4), there exists $z \in \mathbb{F}$ such that $z \approx [(r_n)]$. We show that z is the greatest element in L , or the least element in U .

Seeking a contradiction, suppose that (L, U) is a gap. We need to consider four possibilities resulting from a combination of the following conditions: (1) $z \in L$, (2) $z \in U$, (a) the set of odd numbers belongs to \mathcal{U} , (b) the set of even numbers belongs to \mathcal{U} .

(Ad 1a.) Suppose $z \in L$ and the set of odd numbers belongs to \mathcal{U} . There exists $x \in L$, such that $z < x$, for (L, U) is a gap. Set

$$(2) \quad \theta = \frac{x - z}{2}.$$

Since $z \approx [(r_n)]$, it follows that

$$\{n \in \mathbb{N} \mid |r_n - z| < \theta\} \in \mathcal{U}.$$

Put

$$A = \{n \in \mathbb{N} : n \text{ is odd}\} \cap \{n \in \mathbb{N} \mid |r_n - z| < \theta\} \cap \{n \in \mathbb{N} : \frac{1}{n} < \theta\}.$$

First, $A \in \mathcal{U}$. Second, if $k \in A$, then $r_k - z < \theta$; in consequence

$$(3) \quad r_k < x - \theta.$$

By (2), we have $r_k < x$. Thus $r_k \in L$.

On the other hand, it follows from (1) that

$$(4) \quad r_{k+1} - r_k < \frac{1}{k} < \theta.$$

By adding inequalities (3) and (4) we obtain $r_{k+1} < x$. Hence $r_{k+1} \in L$. But $k+1$ is even, so $r_{k+1} \in U$, contrary to the assumption $L \cap U = \emptyset$.

(Ad 1b.) In the same manner, suppose $z \in L$ and the set of even numbers belongs to \mathcal{U} . Let $x \in L$ be such that $z < x$; set $\theta = \frac{x-z}{2}$. Put

$$A = \{n \in \mathbb{N} : n \text{ is even}\} \cap \{n \in \mathbb{N} \mid |r_n - z| < \theta\} \cap \{n \in \mathbb{N} : \frac{1}{n} < \theta\}.$$

¹⁶For another proof of proposition 3 see [13], however, it implicitly relies on the assumption that \mathcal{F} is Archimedean.

Let $k \in A$. As before, $r_k < x$, which gives $r_k \in L$. Since k is even, the term r_k belongs to U . Thus $r_k \in L \cap U$, a contradiction.

The same reasoning applies to cases (2a), (2b). \square

5.3. Let $r \in \mathbb{R}$. In our proof of FTA, we will also need these simple facts:¹⁷

If $r > 0$, then for every $\varepsilon \in \Omega$ the relation obtains $r + \varepsilon > 0$.

If $r < 0$, then for every $\varepsilon \in \Omega$ the relation obtains $r + \varepsilon < 0$.

For the proof of the first fact, note that if $r > 0$, then for $\varepsilon \in \Omega$ holds $|\varepsilon| < r/2$. Hence $r + \varepsilon > r/2$, and $r + \varepsilon > 0$.

These two facts imply the third result:

If $x \approx y$ and $x \cdot y < 0$, then $x, y \in \mathbb{L}$ and the standard part of x is equal to 0,

$$(5) \quad (\forall x, y \in \mathbb{R}^*)(x \approx y, x \cdot y < 0 \Rightarrow x^o = 0).$$

6. Polynomials. Let $f \in \mathbb{R}[x]$ be a real polynomial,

$$f(x) = a_0 + a_1x + \dots + a_mx^m, \quad \text{where } a_i \in \mathbb{R}.$$

By f^* we mean a hyperreal polynomial $f^* : \mathbb{R}^* \mapsto \mathbb{R}^*$ with real coefficients a_i ,

$$f^*(x) = a_0 + a_1x + \dots + a_mx^m,$$

defined by

$$(6) \quad f^*([(r_n)]) = [(f(r_1), f(r_2), \dots)].$$

If $r \in \mathbb{R}$, then $f^*(r) = [(f(r), f(r), \dots)]$. Since we identify real number r with hyperreal r^* , the equality $f^*(r) = f(r)$ obtains.

Lemma. Let $f \in \mathbb{R}[x]$ and $a \in \mathbb{L}$. If $a \approx b$, then $f^*(a) \approx f^*(b)$.

PROOF. Set

$$f^*(x) = a_0 + a_1x + \dots + a_mx^m, \quad a_i \in \mathbb{R}.$$

Let r be the standard part of a , that is $r = a^o$. Thus $r \approx a$, and for some $\varepsilon \in \Omega$ we have $a = r + \varepsilon$.

Now, for the real number r , the following equalities hold

$$\begin{aligned} f^*(r + \varepsilon) &= a_0 + a_1(r + \varepsilon) + \dots + a_m(r + \varepsilon)^m \\ &= f^*(r) + \varepsilon \cdot w(a_1, \dots, a_m, r, \varepsilon), \end{aligned}$$

where $w(a_1, \dots, a_m, r, \varepsilon) \in \mathbb{L}$. Since infinitesimals form an ideal of the ring \mathbb{L} , the hyperreal number $\varepsilon \cdot w(a_1, \dots, a_m, r, \varepsilon)$ belongs to Ω . Hence

¹⁷One can consider these facts a nonstandard counterpart of the rule (SR), given above in section 2.

$f^*(r + \varepsilon) - f^*(r) \in \Omega$. The identification $f^*(r) = f(r)$ clearly forces $f^*(r + \varepsilon) \approx f(r)$. Since $a = r + \varepsilon$, we obtain

$$f^*(a) \approx f(r).$$

If $a \approx b$, then via the transitivity of the relation *is infinitely close*, we get $r \approx b$. By the uniqueness of the standard part, we also have $r = b^\circ$. The reasoning applied to the pair r, a works for the pair r, b too, thus

$$f^*(b) \approx f(r).$$

Finally, once more applying the transitivity of the relation *is infinitely close*, we have

$$f^*(a) \approx f^*(b).$$

□

One can consider this lemma as a nonstandard counterpart of the standard claim that a real polynomial is a continuous function. Indeed, Birkhoff and Mac Lane, in Chapter 4, entitled *Real Numbers*, of their [3] provide a proof for this claim which applies the $\varepsilon - \delta$ technique. In the next chapter, introducing the proof of FTA they write: “Many proofs of this celebrated theorem are known. All proofs involve nonalgebraic concepts like those introduced in Chap 4” [3, p. 114].

We consider the proof of the following proposition purely algebraic. By Proposition 1, this proposition is equivalent to FTA.

Proposition 4. Let $f \in \mathbb{R}[x]$ and $a, b \in \mathbb{R}$. If $f(a) \cdot f(b) < 0$, then for some $c \in (a, b)$ holds $f(c) = 0$.

PROOF. Obviously, we can take $a = 0$, $b = 1$. Suppose $f(0) < 0$ and $f(1) > 0$. By (6) the same relation obtains in the realm of hyperreals, that is $f^*(0) < 0$ and $f^*(1) > 0$.

Let $N = [(n_j)] = [(n_1, n_2, \dots)]$ be an infinitely large hyperinteger; we can take, for example, $N = [(n)]$. Set

$$I_N = \{K/N : 0 \leq K \leq N\} = \left\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N-1}{N}, 1\right\}.$$

In a similar way we define sets I_{n_j} ,

$$I_{n_j} = \{k/n_j : 0 \leq k \leq n_j\} = \left\{0, \frac{1}{n_j}, \frac{2}{n_j}, \dots, \frac{n_j-1}{n_j}, 1\right\}.$$

However, while sets I_{n_j} are finite, the set I_N is infinite; in fact, it has cardinality continuum.¹⁸

The image of I_{n_j} under the map f is a finite set

$$f(I_{n_j}) = \{f(0), f(1/n_j), \dots, f(1)\}.$$

¹⁸The set I_N is usually called a *hyperfine grid*.

If for some k we have $f(k/n_j) = 0$, then the proof is done. Thus, we can assume the elements of $f(I_{n_j})$ are either negative or positive. Let k_j , where $0 \leq k_j < n_j$, be the first integer such that $f(k/n_j) < 0$ and $f((k+1)/n_j) > 0$,

$$(7) \quad f(k_j/n_j) < 0, \quad f((k_j+1)/n_j) > 0.$$

We can always find such k_j , for the sets I_{n_j} are finite.

Put $K = [(k_j)]$. Since $0 \leq k_j < n_j$, hyperinteger K satisfies the inequalities $0 \leq K < N$. Moreover, $K+1 = [(k_j+1)]$.

By (6) and (7) we have

$$(8) \quad f^*(K/N) < 0, \quad f^*((K+1)/N) > 0.$$

Number K/N is a limited hyperreal. Let c be its standard part, that is $c = (K/N)^o$.

Since N is infinitely large, the element $1/N$ belongs to Ω . Thus $K/N \approx (K+1)/N$, and via the transitivity of the relation *is infinitely close* we obtain

$$(9) \quad K/N \approx c \approx (K+1)/N.$$

Next, by the lemma, it follows that

$$(10) \quad f^*(K/N) \approx f(c) \approx f^*((K+1)/N).$$

Now we come to the final part of the proof. By (10) hyperreals $f^*(K/N)$ and $f^*((K+1)/N)$ are infinitely close; by (8) they have opposite signs. Then, by (5), the standard part of $f^*(K/N)$ is equal to 0, that is

$$(f^*(K/N))^o = 0.$$

On the other hand, by Proposition 3, there exists one and only one standard real number infinitely close to $f^*(K/N)$; by (10) it is the number $f(c)$, that is

$$(f^*(K/N))^o = f(c).$$

Hence then $f(c) = 0$.

To end the proof, observe that the hyperreal K/N belongs to the segment $(0, 1)$, so its standard part, c , lies in the segment $[0, 1]$. Since $f(0) \cdot f(1) < 0$, the real number c equals neither 0 nor 1. As a result, c lies in the segment $(0, 1)$. \square

Following, we show that a polynomial with hyperreal coefficients has the intermediate value property.

Proposition 5. Let $f \in \mathbb{R}^*[x]$ and $a, b \in \mathbb{R}^*$. If $f(a) \cdot f(b) < 0$, then for some $c \in (a, b)$ holds $f(c) = 0$.

PROOF. Let a hyperreal polynomial

$$f(x) = A_0 + A_1x + \dots + A_mx^m, \quad A_i \in \mathbb{R}^*, \quad m \in \mathbb{N},$$

be such that $f(a) < 0$ and $f(b) > 0$, with $a = [(a_n)]$ and $b = [(b_n)]$. Suppose $A_i = [(r_{i,n})]$, where $0 \leq i \leq m$. The hyperreal polynomial f is accompanied by a family of real polynomials $f_n \in \mathbb{R}[x]$, where

$$f_n(x) = r_{0,n} + r_{1,n}x + \dots + r_{m,n}x^m, \quad n \in \mathbb{N}.$$

Indeed, what we really have is a function $f = [f_n]$, where $[f_n] : \mathbb{R}^* \mapsto \mathbb{R}^*$ is defined by

$$(11) \quad [f_n]([(d_n)]) = [(f_1(d_1), f_2(d_2), \dots)].$$

Set

$$I = \{n \in \mathbb{N} : f_n(a_n) < 0\}, \quad J = \{n \in \mathbb{N} : f_n(b_n) > 0\}.$$

Since $I, J \in \mathcal{U}$, the intersection $I \cap J$ also belongs to \mathcal{U} . We can take into consideration only real polynomials f_n with indexes $n \in I \cap J$. Thus, for any f_n , where $n \in I \cap J$, we have

$$f_n(a_n) \cdot f_n(b_n) < 0.$$

By Proposition 4, there exists $c_n \in (a_n, b_n)$ such that $f_n(c_n) = 0$, for $n \in I \cap J$. For indices $n \in \mathbb{N} \setminus I \cap J$, we can take $c_n = 0$.¹⁹ Put $c = [(c_n)]$. By (11), the equality $f(c) = 0$ holds.

Since

$$a_n < c_n < b_n, \quad \text{with } n \in I \cap J,$$

we have the inequalities

$$[(a_n)] < [(c_n)] < [(b_n)].$$

Hence $c \in (a, b)$. □

To end the paper, by Propositions (1) and (5), we obtain

Corollary. The hyperreal number field is real closed and the field $\mathbb{R}^*(\sqrt{-1})$ is algebraically closed.

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¹⁹The equality $[(r_n)] = [(s_n)]$ holds if $r_n = s_n$ for indices n , which belong to some element of the ultrafilter \mathcal{U} . Thus, defining a hyperreal $[(c_n)]$ only those terms c_n matter, of which indices n belong to same element of the ultrafilter \mathcal{U} .

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