Nonstandard Analysis from a philosophical point of view

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1. Nonstandard Analysis is develop on the same grounds that standard, real analysis, e.g. classical logic, axioms for the real numbers field, axiom of choice, and it is in this sense a part of classical mathematics. It is its basic structure, i.e. non-Archimedean field of hyperreals, and **techniques** such as transfer principle, S-continuity, hyperfinite sets and Loeb measures that make it nonstandard. **Problems** it deals with, those it shares with standard analysis, make it to be still analysis rather than a new branch of mathematics.

In my talk I sketch a project in philosophy of mathematics (see [2], [3],[14]) that is designed to investigate the notions of a mathematical problem and a mathematical technique. It is well-known that *the same* theorem, eg. triangle proportionality theorem, Pythagoream theorem (see [10], VI. 2, VI.8), can be proved with different techniques, eg. theory of proportion (see [10], Book V) or the arithmetic of real numbers (see [4]). However, different techniques usually refer to different mathematical structures. Since there is some common ground between standard and nonstandard analysis, I choose mathematical analysis to develop a notion of a mathematical problem and a mathematical technique. I discuss [8] to present a problem that is not comprehended in any axiomatic reconstruction of mathematics. Next, I present a brand new technique of nonstandard analysis, namely that of hyperfinite sets. Finally, I address the question of a mathematical technique itself.

In the talk basic knowledge of nonstandard analysis is assumed (see point 5 below).

2. *Mathematics over Metaphysics.* In [8] Dedekind introduces „a real definition of the **essence of continuity**" that could form a sufficient basis for „a rigorous exposition of differential calculus". It is also believed that opposed to continuity is discreteness (see [1],[13]). Nowadays Dedekind's continuity is just a characterization of a totally ordered set. In this context, discrete, as opposed to continuous, means discrete order. Next to continuous order, there are other notions of continuity in use in mathematics, e.g. Dedekind complete ordered field or topological field (to mention only those that characterize an algebraic field). Mathematics also provides a more general meaning of discrete: in topology, discrete, as opposed to connected, could be rendered as totally disconnected space.

The field of hyperreals is not Dedekind continuous, and it is also a totally disconnected topological space, so, in a sense, it is a discrete space. In spite of this, within the framework of Nonstadard Analysis basic theorems of standard analysis can be proved. Comparing standard and nonstandard analysis I show that beyond mathematical rules for defining numbers such as Dedekind cut, Cauchy completeness, standard part theorem or hyperfinite sum there is nothing like *the linear continuum*.

3. *Finite-Infinite-Hyperfinite.* In classical mathematics the set of natural numbers $\mathbb{N}$ forms a standard measure of infinity: a set A is finite iff there is a bijection between it a some natural number n, otherwise it is infinite. This Cantorian approach focuses on the cardinality of a set. However, one can take into account the well-known properties of finite sets, namely: (1) a subset of a finite set is finite, (2) a finite and totally ordered set has a greatest and a least element, (3) if A, B are finite then $\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}} - \overline{\overline{A \cap B}}$. **Hyperfinite sets**, being either finite or denumerable in Cantor's sense, share with standard finite sets (in a sense clarified below) these properties. I present some arguments of Nonstandard Analysis that make use of these properties and their standard analysis counterparts (eg. Riemann integral, Lebesgue measure) that refer to the notion of limit and, in consequence, to the axiom of continuity.

4. *Mathematical techniques over logic.* Philosophically motivated programs to reconstruct analysis on different **grounds** than those provided by real analysis are based on a tacit assumption that there is some ground structure of analysis, usually called *real numbers* (see [5],[11]). As a result they mimic basic real analysis concepts (eg. ordered field, sequence and limit, continuity of a function) to develop but a new branch of mathematics. Since the field of rationals is a common ground between standard, constructivist (see [5], p. 42) and intuitionist (see [11], p. 16) analysis I present the ordered field of rational numbers just as a mathematical technique rather than a construction.

5. Basic facts and definitions (see [6],[7],[9],[12]). Let $(\mathbb{R}, +, \cdot, 0, 1, <)$ be the field of real numbers, $\mathcal{F}$ – a nonprincipal ultrafilter on $\mathbb{N}$. The relation defined by

$$(r_n) \equiv (s_n) \leftrightarrow_{df} \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{F}$$

is an equivalence relation on the set $\mathbb{R}^{\mathbb{N}}$. The set of hyperreals $\mathbb{R}^*$ is the quotient set $\mathbb{R}^* =_{df} \mathbb{R}^{\mathbb{N}}/_{\equiv}$.

Addition, multiplication and order of hyperreals are defined by

$$[(r_n)] \oplus [(s_n)] =_{df} [(r_n + s_n)], \qquad [(r_n)] \otimes [(s_n)] =_{df} [(r_n \cdot s_n)],$$

$$[(r_n)] \prec [(s_n)] \leftrightarrow_{df} \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{F}.$$

The standard real number r is identified with equivalence class r^* of the constant sequence $(r, r, ...)$, i.e. $r^* =_{df} [(r, r, ...)]$.

Theorem $(\mathbb{R}^*, \oplus, \otimes, 0^*, 1^*, \prec)$ is a non-Archimedean, real closed field.

The set of infinitisimal hyperreals Ω is defined by

$$x \in \Omega \leftrightarrow_{df} \forall \theta \in \mathbb{R}_+[\ |x| \prec \theta^*\ ].$$

We say that x is infinitely close to y, $x \approx y$, iff $x - y \in \Omega$.

The set of limited hyperreals $\mathcal{O}$ is defined by

$$x \in \mathcal{O} \leftrightarrow_{df} \exists \theta \in \mathbb{R}_+[\ |x| \prec \theta^*\ ].$$

Standard Part Theorem: $\forall x \in \mathcal{O} \exists! r \in \mathbb{R}[\ r^* \approx x\ ]$.

The standard part of a limited hyperreal x is denoted by ${}^o x$, i.e. ${}^o x = r$.

The set of hypernaturals $\mathbb{N}^*$ is defined by

$$[(n_j)] \in \mathbb{N}^* \leftrightarrow_{df} \{j \in \mathbb{N} : n_j \in \mathbb{N}\} \in \mathcal{F}.$$

The set of infinite hypernaturals $\mathbb{N}_\infty$ is defined by $\mathbb{N}_\infty =_{df} \mathbb{N}^* \setminus \{n^* : n \in \mathbb{N}\}$.

Let $(s_n)_{n \in \mathbb{N}}$ be a sequence of reals. Then an extension of $(s_n)_{n \in \mathbb{N}}$ to a hypersequence $(s^*_K)_{K \in \mathbb{N}^*}$ is defined by

$$s^*_K =_{df} [(s_{k_j})] = [(s_{k_1}, s_{k_2}, ...)], \quad \text{where} \quad K = [(k_j)] = [(k_1, k_2, ...)].$$

Basic Theorem Let (s_n) be a sequence of real numbers, let $a \in \mathbb{R}$. Then

$$\lim_{n \to \infty} s_n = a \leftrightarrow \forall K \in \mathbb{N}_\infty [s^*_K \approx a^*].$$

Let $\{H_n\}$ be a sequence of subsets of $\mathbb{R}$. An internal set $[H_n]$ is a subset of $\mathbb{R}^*$ defined by

$$[(r_n)] \in [H_n] \leftrightarrow_{df} \{n \in \mathbb{N} : r_n \in H_n\} \in \mathcal{F}.$$

When $\{n \in \mathbb{N} : \ H_n \text{ is finite}\} \in \mathcal{F}$, then $[H_n]$ is called hyperfinite. When $H_n = A$, for all n, then the set $[H_n] = [A, A, ...\ ]$ is denoted by A^*, thus $\mathbb{N}^* = [\mathbb{N}, \mathbb{N}, ...\ ]$, $\mathbb{Z}^* = [\mathbb{Z}, \mathbb{Z}, ...\ ]$, and $(a,b)^* = [(a,b), (a,b), ...\ ]$, for $a, b \in \mathbb{R}$.

Theorem: (1) Any internal set is finite or uncountable.
(2) An internal subset of a hyperfinite set is hyperfinite.
(3) Any hyperfinite set has a greatest and a least element.
(4) The union and intersection of any two hyperfinite sets F and G are hyperfinite, with internal cardinality $|F \cup G| = |F| \oplus |G| - |F \cap G|$.

Let f_n be a sequence of real functions such that $f_n : A_n \mapsto \mathbb{R}$. An internal function $[f_n] : [A_n] \mapsto \mathbb{R}^*$ is defined by

$$[f_n]([(r_n)] =_{df} [(f_n(r_n)].$$

The **hyperfinite sum** of a hyperfinite function $[f_n]$ over a hyperfinite set $[H_n]$ is a hyperreal number defined by

$$\sum_{a \in [H_n]} [f_n](a) =_{df} [(\sum_{a \in H_n} f_n(a))].$$

Let $N \in \mathbb{N}_\infty$, the hyperfinite time line is the hyperfinite set

$$T = \{\frac{k}{N} : \ k \in Z^*, \ -N^2 \preceq k \preceq N^2\}.$$

Let $\mathcal{A}$ be the set of all internal subsets of T, i.e. $\mathcal{A} = \{A \subset T : \ A \text{ is internal}\}$. $\mathcal{A}$ is an algebra of sets. Let μ be the *counting* measure on $\mathcal{A}$ defined by

$$\mu(A) = \frac{|A|}{N}.$$

A real valued map ${}^o\mu : \mathcal{A} \mapsto [0, \infty]$, defined by

$${}^o\mu(A) = \begin{cases} {}^0(\mu(A)), & \text{if } \mu(A) \text{ is limited} \\ \infty, & \text{otherwise} \end{cases}$$

is additive and for any sequence of pairwise disjoint sets $(A_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ holds

$$\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A} \rightarrow {}^o\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}}{}^o\mu(A_n).$$

Theorem There is a unique extension of ${}^o\mu$ to the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. The completion of this measure is the Loab measure μ_L and the completion of $\sigma(\mathcal{A})$ is the Loab σ-algebra $L(\mathcal{A})$.
Theorem Let $\mathcal{B}$ be the set $\mathcal{B} = \{B \subset \mathbb{R} : \ st^{-1}(B) \in L(\mathcal{A})\}$, where $st^{-1}(B) = \{t \in T : \ {}^o t \in B\}$. Then a measure λ on $\mathcal{B}$, defined by

$$\lambda(B) = \mu_L(st^{-1}(B)),$$

is the Lebesgue measure.
Corollary For any $a, b \in \mathbb{R}$, with $a < b$,

$$\mu_L(\{t \in T : \ a^* \prec t \prec b^*\}) =^o \mu\Big(\frac{|T \cap (a,b)^*|}{N}\Big) = b - a.$$

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