

Interpreting the Infinitesimal Mathematics of Leibniz and Euler

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Abstract We apply Benacerraf's distinction between mathematical *ontology* and mathematical *practice* (or the *structures* mathematicians use in practice) to examine contrasting interpretations of infinitesimal mathematics of the seventeenth and eighteenth century, in the work of Bos, Ferraro, Laugwitz, and others. We detect Weierstrass's ghost behind some of the received historiography on Euler's infinitesimal mathematics, as when Ferraro proposes to *understand* Euler in terms of a Weierstrassian notion of limit and Fraser declares classical analysis to be a "primary point of reference for understanding the eighteenth-century theories." Meanwhile, scholars like Bos and Laugwitz seek to explore Eulerian *methodology*, *practice*, and *procedures* in a way more faithful to Euler's own. Euler's use of infinite integers and the associated infinite products are analyzed in the context of his infinite product decomposition for the sine function. Euler's *principle of cancellation* is compared to the Leibnizian *transcendental law of homogeneity*. The Leibnizian *law of continuity* similarly finds echoes in Euler. We argue that Ferraro's assumption that Euler worked with a *classical* notion of quantity is symptomatic of a post-

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Weierstrassian placement of Euler in the Archimedean track for the development of analysis, as well as a blurring of the distinction between the dual tracks noted by Bos. Interpreting Euler in an Archimedean conceptual framework obscures important aspects of Euler's work. Such a framework is profitably replaced by a syntactically more versatile modern infinitesimal framework that provides better proxies for his inferential moves.

Keywords Archimedean axiom · Infinite product · Infinitesimal · Law of continuity · Law of homogeneity · Principle of cancellation · Procedure · Standard part principle · Ontology · Mathematical practice · Euler · Leibniz

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1 Introduction

This text is part of a broader project of re-appraisal of the Leibniz–Euler–Cauchy tradition in infinitesimal mathematics that Weierstrass and his followers broke with around 1870.

In the case of Cauchy, our task is made easier by the largely traditionalist scholars G. Schubring and G. Ferraro. Thus, Schubring distanced himself from the Boyer–Grabiner line on Cauchy as the one *who gave you the epsilon* in the following terms: “I am criticizing historiographical approaches like that of Judith Grabiner where one sees epsilon-delta already realized in Cauchy” (Schubring 2016, Section 3). Ferraro goes even further and declares: “Cauchy uses infinitesimal neighborhoods of x in a decisive way... Infinitesimals are not thought as a mere *façon de parler*, but they are conceived as numbers, though a theory of infinitesimal numbers is lacking” (Ferraro 2008, 354). Ferraro's comment is remarkable for two reasons:

- it displays a clear grasp of the procedure versus ontology distinction (see below Sect. 2.4);
- it is a striking recognition of the *bona fide* nature of Cauchy's infinitesimals that is a clear break with Boyer–Grabiner.

Ferraro's comment is influenced by Laugwitz's perceptive analysis of Cauchy's sum theorem in Laugwitz (1987), a paper cited several times on Ferraro (2008, 354). For further details on Cauchy see the articles by Błaszczyk et al. (2013, 2016).

In this article, we propose a re-evaluation of Euler's and, to an extent, Leibniz's work in analysis. We will present our argument in four stages of increasing degree of controversy, so that readers may benefit from the text even if they don't agree with all of its conclusions.

- (1) We argue that Euler's procedures in analysis are best proxified in modern infinitesimal frameworks rather than in the received modern Archimedean ones, by showing how important aspects of his work have been underappreciated or even denigrated because inappropriate conceptual frameworks are being applied to interpret his work. To appreciate properly Euler's work, one needs to abandon extraneous *ontological* matters such as the continuum being punctiform (i.e., made out of points) or nonpunctiform, and focus on the *procedural* issues of Euler's actual mathematical practice.
- (2) One underappreciated aspect of Euler's work in analysis is its affinity to Leibniz's. A number of Eulerian procedures are consonant with those found in Leibniz, such as

the law of continuity (governing the passage from an Archimedean continuum to an infinitesimal-enriched continuum) and the transcendental law of homogeneity (governing the passage from an infinitesimal-enriched continuum back to an Archimedean continuum). This is consistent with the teacher–student lineage from Leibniz to Johann Bernoulli to Euler.

- (3) Leibniz wrote in 1695 that his infinitesimals violate the property expressed by Euclid’s Definition V.4 (see Leibniz 1695, 288).¹ This axiom is a variant of what is known today as the Archimedean property. Thus, Leibnizian infinitesimals violate the Archimedean property when compared to other quantities.
- (4) Our reading is at odds with the *syncategorematic* interpretation elaborated in (Ishiguro 1990, Chapter 5), Arthur (2008), and elsewhere. Ishiguro, Arthur, and others maintain that Leibniz’s continuum was Archimedean, and that his infinitesimals do not *designate* and are *logical fictions* in the sense of Russell. The leap by Ishiguro (and her followers) from infinitesimals being *fictions* to their being *logical fictions* is a non-sequitur analyzed by Katz and Sherry (2013) and Sherry and Katz (2014). Arthur’s interpretation was also challenged in Tho (2012). The fictions in question are *pure* rather than *logical*, meaning that they do *designate* insofar as our symbolism allows us to think about infinitesimals. This is consistent with interpretations of Leibniz by Bos (1974) and Jessep (2015) (see Sect. 3.2 for a discussion of Jessep’s analysis). Euler similarly works explicitly with infinite and infinitesimal numbers rather than some kind of paraphrase thereof in terms of proto-Weierstrassian hidden quantifiers.

In “Appendix”, we examine the mathematical details of the Eulerian procedures in the context of his proof of the infinite product decomposition for the sine function and related results.

In addition to Robinson’s framework, other modern theories of infinitesimals are also available as possible frameworks for the interpretation of Euler’s procedures, such as Synthetic Differential Geometry (Kock 2006; Bell 2008) and Internal Set Theory (Nelson 1977; Kanovei and Reeken 2004). See also Nowik and Katz (2015) as well as Kanovei et al. (2016). Previous studies of the history of infinitesimal mathematics include Katz and Katz (2011), Borovik and Katz (2012), Bair et al. (2013), Katz et al. (2013), Carroll et al. (2013), Bascelli et al. (2014), Kanovei et al. (2015).

2 Historiography

It is a subject of contention among scholars whether science (including mathematics) develops continuously or by discontinuous leaps. The idea of paradigm shift by Kuhn (1962) is the most famous instance of the discontinuous approach. The discontinuous case is harder to make for mathematics than for the physical sciences: we gave up on phlogiston and caloric theory, but we still use the Pythagorean theorem and l’Hôpital’s rule.

¹ Actually Leibniz referred to V.5; in some editions of the *Elements* this Definition does appear as V.5. Thus, Euclid (1660) as translated by Barrow in 1660 provides the following definition in V.V (the notation “V.V” is from Barrow’s translation): *Those numbers are said to have a ratio betwixt them, which being multiplied may exceed one the other.*

2.1 Continuity and Discontinuity

We argue that the continuous versus discontinuous dichotomy is relevant to understanding some of the current debates in interpreting classical infinitesimalists like Leibniz and Euler. Thus, A. Robinson argued for continuity between the Leibnizian framework and his own, while H. Bos rejected Robinson's contention in the following terms:

... the most essential part of non-standard analysis, namely the proof of the existence of the entities it deals with, was entirely absent in the Leibnizian infinitesimal analysis, and this constitutes, in my view, so fundamental a difference between the theories that the Leibnizian analysis cannot be called an early form, or a precursor, of non-standard analysis. (Bos 1974, 83)

Of course, many scholars reject continuity not merely between Robinson's framework and historical infinitesimals, but also between the received modern mathematical frameworks and historical infinitesimals. A case in point is Ferraro's treatment of an infinitesimal calculation found in Euler (1730, 11f.). Here Euler sought the value of the ratio $(1 - x^{g/(f+g)})/g$ for $f = 1$ and $g = 0$ by applying l'Hôpital's rule to $\frac{1-x^c}{z}$. Ferraro proceeds to present the problem "from a modern perspective" by analyzing the function $f(z) = \frac{1-x^z}{z}$ and its behavior near $z = 0$ in the following terms:

From the modern perspective, the problem of extending the function $f(z) = \frac{1-x^z}{z}$ in a continuous way means that... the domain D of $f(z)$ has a point of accumulation at 0 so that we can attempt to calculate the limit as $z \rightarrow 0$, where by $\lambda = \lim_{z \rightarrow c} f(z)$ [the c in Ferraro's formula needs to be replaced by 0] we mean: given any $\varepsilon > 0$ there exists a $\delta > 0$ such that if z belongs to D and $|z| < \delta$ then $|f(z) - \lambda| < \varepsilon$; ... This procedure is *substantially meaningless* for Euler. (Ferraro 2004, 46, emphasis added)

Ferraro's concluding remarks concerning "substantially meaningless" procedures place him in the discontinuity camp.

While there is a great deal of truth in the discontinuous position, particularly with regard to currently prevalent ontological frameworks (set-theoretic or category-theoretic), we will argue for a limited reading of the history of analysis from the perspective of *continuous* development in the following sense. As we analyze the history of analysis since the seventeenth century, we note stark differences among the objects with which mathematicians reason; there are for example no *sets* as explicit mathematical objects in Leibniz or Euler. On the other hand, there are important continuities in the principles which guide the inferences that they draw; for example, Leibniz's *transcendental law of homogeneity*, Euler's *principle of cancellation*, and the *standard part principle* exploited in analysis over a hyperreal extension $\mathbb{R} \subseteq {}^*\mathbb{R}$.

The crucial distinction here is between practice and ontology, as we detail below in Sect. 2.4. We will argue that there is a historical continuity in mathematical practice but discontinuity in mathematical ontology. More specifically, the set-theoretic semantics that currently holds sway is a discontinuity with respect to the historical evolution of mathematics. Scholars at times acknowledge the distinction in relation to their own work, as when Ferraro speaks about the *intensional* nature of the entities in Euler in (Ferraro 2004, 44) and the *syntactic* nature of algebraic and analytic operations (Ferraro 2008, 203), but not always when it comes to passing judgment on Laugwitz's work; see Sect. 2.2.

2.2 Procedures and Proxies

In the case of Euler, we will examine philosophical issues of interpretation of infinitesimal mathematics (more specifically, the use of infinitesimals and infinite integers) and seek to explore the roots of the current situation in Euler scholarship, which seems to be something of a dialog of the deaf between competing approaches. Some aspects of Euler's work in analysis were formalized in terms of modern infinitesimal theories by Laugwitz, McKinzie, Tuckey, and others. Referring to the latter, G. Ferraro claims that "one can see in operation in their writings a conception of mathematics which is *quite extraneous* to that of Euler" (Ferraro 2004, 51, emphasis added). Ferraro concludes that "the attempt to specify Euler's notions by applying modern concepts is only possible if elements are used which are *essentially alien* to them, and thus Eulerian mathematics is transformed into something wholly different" (Ferraro 2004, 51–52, emphasis added).

Now *quite extraneous* and *essentially alien* are strong criticisms. The vagueness of the phrase "to specify Euler's notions by applying modern concepts" makes it difficult to evaluate Ferraro's claim here. If *specification* amounts to bringing to light tacit assumptions in Euler's reasoning, then it is hard to see why Ferraro uses such harsh language.

We find a different attitude in P. Reeder's approach to Euler. Reeder writes:

I aim to reformulate a pair of proofs from [Euler's] *Introductio* using concepts and techniques from Abraham Robinson's celebrated non-standard analysis (NSA). I will specifically examine Euler's proof of the Euler formula and his proof of the divergence of the harmonic series. Both of these results have been proved in subsequent centuries using epsilon-delta (standard epsilon-delta) arguments. The epsilon-delta arguments differ significantly from Euler's original proofs. (Reeder 2012, 6)

Reeder concludes that "NSA possesses the tools to provide appropriate proxies of the inferential moves found in the *Introductio*." Reeder finds significant similarities between some of Euler's proofs and proofs in a hyperreal framework. Such similarities are missing when one compares Euler's proofs to proofs in the ϵ, δ tradition. We take this to mean that Euler's conception has more in common with the syntactic resources available in a modern infinitesimal tradition than in the ϵ, δ tradition.

Scholars thus appear to disagree sharply as to the relevance of modern theories to Euler's mathematics, and as to the possibility of meaningfully reformulating Euler's infinitesimal mathematics in terms of modern theories.

2.3 Precalculus or Analysis?

Having mentioned Euler's *Introductio*, we would like to clarify a point concerning the nature of this book. Blanton writes in his introduction that "the work is strictly precalculus" (Euler 1988, xii). Is this an accurate description of the book? It is worth keeping the following points in mind.

- (1) The algebraic nature of the *Introductio* was mirrored 70 years later by Cauchy's *Cours d'Analyse*, which was subtitled *Analyse Algèbrique*. Laugwitz noted in fact that *Cours d'Analyse* was modeled on Euler's *Introductio* (Laugwitz 1999, 52).
- (2) There may not be much material related to differentiation in *Introductio*, but series are dealt with extensively. Series certainly being part of analysis, it seems more reasonable to describe *Introductio* as *analysis* than precalculus.

- (3) Infinitesimals in *Introductio* and differentials in *Institutiones* are arguably of similar nature. Leibniz already thought of differentials as infinitesimals, as did Johann Bernoulli. There is little reason to assume otherwise as far as Euler is concerned, particularly since he viewed all his analysis books as a unified whole.

To elaborate further on item (3), note that Euler writes in his *Institutionum calculi integralis* as follows: “In calculo differentiali iam notavi, quaestionem de differentialibus non absolute sed relative esse intelligendam, ita ut, si y fuerit function quaecunque ipsius x , non tam ipsum eius differentiale dy , quam eius ratio ad differentiale dx sit definienda” Euler (1768–1770, 6). This can be translated as follows: “Now in differential calculus I have observed that an investigation of differentiation is to be understood as not absolute but relative; namely, if y is a function of x , what one needs to define is not so much its differential dy itself as its ratio to the differential dx .”

The comment indicates that throughout the period 1748–1768, Euler thinks of infinitesimals and differentials as essentially interchangeable.

2.4 Practice Versus Ontology

In an influential essay “The Relation Between Philosophy of Science and History of Science,” M. Wartofsky argues that historiography of science needs to begin its analysis by mapping out an ontology of the scientific field under investigation. Here ontology is to be understood in a broader sense than merely the ontology of the entities exploited in that particular science—such as numbers, functions, sets, etc., in the case of mathematics—but rather to develop the ontology of mathematics as a scientific theory itself (Wartofsky 1976, 723).

As a modest step in this direction we distinguish between the (historically relative) ontology of the mathematical objects in a certain historical setting, and its procedures, particularly emphasizing the different roles these components play in the history of mathematics. More precisely, our *procedures* are representative of what Wartofsky called the *praxis* characteristic of the mathematics of a certain time period, and our *ontology* takes care of the mathematical objects recognized at that time.

To motivate our adherence to procedural issues, we note that there is nothing wrong in principle with investigating *pure ontology*. However, practically speaking attempts by historians to gain insight into Euler’s ontology (as opposed to procedures) have a tendency, to borrow Joseph Brodsky’s comment in his introduction to Andrei Platonov’s novel *The Foundation Pit*, to *choke on their own subjunctive mode*, as richly illustrated by an *ontological* passage that we quote in Sec. 2.8.

The dichotomy of *mathematical practice* versus *ontology of mathematical entities* has been discussed by a number of authors including W. Quine, who wrote: “Arithmetic is, in this sense, all there is to number: there is no saying absolutely what the numbers are; there is only arithmetic” (Quine 1968, 198).

For our purposes it will be more convenient to rely on Benacerraf’s framework. Benacerraf (1965) pointed out that if observer E learned that the natural numbers “are” the Zermelo ordinals

$$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots,$$

while observer J learned that they are the von Neumann ordinals

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots$$

then, strictly speaking, they are dealing with different things. Nevertheless, observer E's actual mathematical practice (and the mathematical structures he is interested in) is practically the same as observer J's. Hence, different ontologies may underwrite one and the same practice.

For observer E, the entity 0 is *not* an element of the entity 2, while for observer J it is. But for both of them the *relation* $0 < 2$ holds. Benacerraf's point is that although mathematicians carry on their reasoning in terms of some objects or others, the particular objects are not so important as the *relations* among those objects. The relations may be the same, even though the objects are different.

We would extend this insight beyond differences in set-theoretic foundations and argue that even though Euler reasons about quantities and Robinson reasons about sets (or types), they both agree, for example, that $a + dx = a$ for infinitesimal dx in a suitable generalized sense of equality. This is made precise in a hyperreal framework via the standard part principle; see Sect. 3.3 for more details.

This distinction relativizes the import of ontology in understanding mathematical *practice*. A year after the publication of Benacerraf's text, a related distinction was made by Robinson in syntactic/procedural terms:

...the theory of this book ... is presented, naturally, within the framework of contemporary Mathematics, and thus appears to affirm the existence of all sorts of infinitary entities. However, from a formalist point of view we may look at our theory syntactically and may consider that what we have done is to introduce *new deductive procedures* rather than new mathematical entities. (Robinson 1966, 282, emphasis in the original)

In short, we have, on the one hand, the *ontological* issue of giving a foundational account for the *entities*, such as infinitesimals and infinite integers, that classical infinitesimalists may be working with. On the other hand, we have their *procedures*, or *inferential moves*, termed *syntactic* by Robinson. What interests Euler scholars like Laugwitz is not Euler's ontology but the syntactic *procedures* of his mathematical practice. The contention that B-track formalisations (see Sect. 2.5) provide better *proxies* for Euler's procedures and inferential moves than A-track formalisations, is a *methodological* or *instrumentalist* rather than an ontological or foundational matter.

To quote H. Pulte: "Philosophy of science today should offer a more accurate analysis to history of science without giving up its task – not always appreciated by historians – to uncover the basic concepts and *methods* which seem relevant for the understanding of science in question" (Pulte 2012, 184, emphasis added). Lagrange's approach in his 1788 *Mécanique analytique* was remarkably modern in its instrumentalism:

Neither are the metaphysical premises of his mechanics made explicit, nor is there any epistemological justification given for the presumed infallible character of the basic principles of mechanics. This is in striking contrast not only to seventeenth century foundations of mechanics such as that of Descartes, Leibniz, and Newton but also to the approaches of Lagrange's immediate predecessors, Euler, Maupertuis, or d'Alembert ... In short, a century after Newton's *Principia*, Lagrange's textbook can be seen as an attempt to update the mathematical principles of natural philosophy while abandoning the traditional subjects of *philosophia naturalis*. In this *special* sense, the *Mécanique analytique* [sic] is also a striking example of mathematical instrumentalism. (Pulte 1998, 158, emphasis in the original)

Two and a quarter centuries after Lagrange's instrumentalist approach, perhaps a case can be made in favor of a historiography focusing on methodological issues accompanied by an instrumentalist caution concerning metaphysics and/or ontology of mathematical entities

like numbers and quantities. This is in line with Pulte’s insightful comment made in the context of the study of rational mechanics in the eighteenth century: “Euclideanism continues to be the ideal of science, but it becomes a syntactical rather than a semantical concept of science” (Pulte 2012, 192).

2.5 A-track and B-track from Klein to Bos

The sentiment that there have been historically at least two possible approaches to the foundations of analysis, involving dual methodology, has been expressed by a number of authors.

In 1908, Felix Klein described a rivalry of two types of continua in the following terms. Having outlined the developments in real analysis associated with Weierstrass and his followers, Klein pointed out that “The scientific mathematics of today is built upon the series of developments which we have been outlining. But an essentially different conception of infinitesimal calculus has been running parallel with this [conception] through the centuries” (Klein 1932, 214). Such a different conception, according to Klein, “harks back to old metaphysical speculations concerning the structure of the continuum according to which this was made up of ... infinitely small parts” (ibid.). Thus according to Klein there is not one but two separate tracks for the development of analysis:

- (A) the Weierstrassian approach (in the context of an *Archimedean* continuum); and
- (B) the approach with indivisibles and/or infinitesimals (in the context of what could be called a *Bernoullian* continuum).

For additional details on Klein see Sect. 4.3.

A similar distinction can be found in Henk Bos’s seminal 1974 study of Leibnizian methodology. Here Bos argued that *distinct* methodologies, based respectively on (Archimedean) exhaustion and on infinitesimals, are found in the work of seventeenth and eighteenth century giants like Leibniz and Euler:

Leibniz considered two different approaches to the foundations of the calculus; one connected with the classical methods of proof by “exhaustion”, the other in connection with a law of continuity. (Bos 1974, 55)

The first approach mentioned by Bos relies on an “exhaustion” methodology in the context of an Archimedean continuum. Exhaustion methodology is based on proofs by *reductio ad absurdum* and the ancient theory of proportion, which, as is generally thought today, is based on the Archimedean axiom.²

² We note, in the context of Leibniz’s reference to Archimedes, that there are other possible interpretations of the exhaustion method of Archimedes. The received interpretation, developed in Dijksterhuis (1987), is in terms of the limit concept of real analysis. However, Wallis (1685, 280–290) developed a different interpretation in terms of approximation by infinite-sided polygons. The ancient exhaustion method has two components:

- (1) *geometric construction*, consisting of approximation by some *simple* figure, e.g., a polygon or a line built of segments,
- (2) *justification* carried out in the theory of proportion as developed in *Elements* Book V.

In the seventeenth century, mathematicians adopted the first component, and developed alternative justifications. The key feature is the method of exhaustion is the logical structure of its proof, namely *reductio ad absurdum*, rather than the nature of the background continuum. The latter can be Bernoullian, as Wallis’ interpretation shows.

One way of formulating the axiom is to require that every positive number can be added to itself finitely many times to obtain a number greater than one. The adjective *Archimedean* in this sense was introduced by O. Stolz in the 1880s (see Sect. 3.9). We will refer to this type of methodology as the A-methodology.

Concerning the second methodology Bos notes: “According to Leibniz, the use of infinitesimals belongs to this kind of argument” (Bos 1974, 57). We will refer to it as the B-methodology, in an allusion to Johann Bernoulli (whose work formed the basis for l’Hôpital 1696), who, having learned an infinitesimal methodology from Leibniz, never wavered from it.

The Leibnizian laws such as the law of continuity mentioned by Bos in the passage cited above, as well as the transcendental law of homogeneity mentioned in Bos (1974, 33), find close procedural analogs in Euler’s work, and indeed in Robinson’s framework. The transcendental law of homogeneity is discussed in Sect. 3.3 and the law of continuity in Sect. 3.6.

In 2004, Ferraro appeared to disagree with Bos’s *dual track* assessment, and argued for what he termed a “continuous leap” between (A-track) limits and (B-track) infinitesimals in Euler’s work; see Sect. 4.10.

2.6 Mancosu and Hacking

To support our contention that there exist two distinct viable tracks for the development of analysis, we call attention to Mancosu’s critique of Gödel’s heuristic argument for the inevitability of the Cantorian cardinalities as the only plausible theory of the infinite. Gödel’s argued that

the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (e.g. their colors or their distribution in space). (Gödel 1990, 254)

Mancosu argues that recent theories on numerosities undermine Gödel’s assumption. These were developed in Benci and Nasso (2003) as well as Nasso and Forti (2010) and elsewhere.³ Mancosu concludes that

having a different way of counting infinite sets shows that while Gödel gives voice to one plausible intuition about how to generalize ‘number’ to infinite sets there are coherent alternatives. (Mancosu 2009, 638)

Inspired in part by Mancosu (2009), Ian Hacking proposes a distinction between the *butterfly model* and the *Latin model*, namely the contrast between a model of a deterministic biological development of animals like butterflies, as opposed to a model of a contingent historical evolution of languages like Latin. For a further discussion of Hacking’s views see Sect. 5 below.

³ A technical comment on numerosities is in order. A numerosity is a finitely additive measure-like function defined on an algebra of sets, which takes values in the positive half of a non-Archimedean ordered ring. A numerosity is *elementary* if and only if it assigns the value 1 to every singleton in the domain, so that the numerosity of any finite set is then equal to its number of elements. Therefore any elementary numerosity can be viewed as a generalization of the notion of finite quantity. Numerosities are sometimes useful in studies related to Lebesgue-like and similar measures, where they help to “individualize” classically infinite measure values, associating them with concrete infinitely large elements of a chosen non-Archimedean ordered ring or field. As a concept of infinite quantity, numerosities have totally different properties, as well as a totally different field of applications, than the Cantorian cardinals.

2.7 Present-Day Standards

Bos's comment on Robinson cited at the beginning of Sect. 2.1 is not sufficiently sensitive to the dichotomy of practice (or procedures) versus ontology (or foundational account for the entities) as analyzed in Sect. 2.4. Leibnizian procedures exploiting infinitesimals find suitable proxies in the procedures in the hyperreal framework; see Reeder (2013) for a related discussion in the context of Euler. The relevance of such hyperreal proxies is in no way diminished by the fact that set-theoretic *foundations* of the latter ("proof of the existence of the entities," as Bos put it) were obviously as unavailable in the seventeenth century as set-theoretic foundations of the real numbers.

In the context of his discussion of "present-day standards of mathematical rigor", Bos writes:

it is understandable that for mathematicians who believe that these present-day standards are final, nonstandard analysis answers positively the question whether, after all, *Leibniz was right*. (Bos 1974, 82, emphasis added)

The context of the discussion makes it clear that Bos's criticism targets Robinson. If so, Bos's criticism suffers from a strawman fallacy, for Robinson specifically wrote that he did not consider set theory to be the foundation of mathematics. Being a formalist, Robinson did not subscribe to the view attributed to him by Bos that "present-day standards are final." Robinson expressed his position on the status of set theory as follows: "an infinitary framework such as set theory ... cannot be regarded as the ultimate foundation for mathematics" (Robinson 1969, 45), see also Robinson (1966, 281). Furthermore, contrary to Bos's claim, Robinson's goal should not be seen as showing that "Leibniz was right" (see above). Rather, Robinson's goal was to provide hyperreal proxies for the inferential procedures commonly found in Leibniz as well as Euler and Cauchy. Leibniz's procedures, involving as they do infinitesimals and infinite numbers, seem far less puzzling when compared to their B-track hyperreal proxies than from the viewpoint of the traditional A-track frameworks; see Sect. 2.5.

2.8 Higher Ontological Order

We wish to emphasize that we do *not* hold that it is only possible to interpret Euler in terms of modern formalisations of his *procedures*. Discussions of Eulerian *ontology* could potentially be fruitful. Yet some of the existing literature in this direction tends to fall short of a standard of complete lucidity. Thus, Panza quotes Euler to the effect that "Just as from the ideas of individuals the idea of species and genus are formed, so a variable quantity is the genus in which are contained all determined quantities," and proceeds to explicate this as follows:

As constant quantities are determined quantities, this is the same as claiming that a variable quantity is the genus in which are contained all constant quantities. A variable quantity is thus a sort of a formal characterization of quantity as such. Its concept responds to a need for generality, i.e. a need of studying the essential properties of any object of a certain genus, the properties that this object has insofar as it belongs to such a genus. But, according to Euler, this study has to have its own objects. In order to identify these objects, it is necessary to sever these essential properties from any other property that characterizes any object falling under the same genus. If the genus is that of quantities, one has thus to identify some objects

that are not specific (and, a fortiori, particular) quantities and pertain thus to a *higher ontological order* than that to which specific quantities pertain. (Panza 2007, 8–9, emphasis added)

We are somewhat confused by this passage which seems to be ontological in nature. Since ontology is not our primary concern here (see Sect. 2.4), we will merely propose further investigation into the nature of variable quantities.

3 Our Reading of Leibniz and Euler

The book *Introductio in Analysin Infinitorum* (Euler 1748) contains remarkable calculations carried out in a framework where the basic algebraic operations are applied to infinitely small and infinitely large quantities.

3.1 Exponential Function

In Chapter 7 on exponentials and logarithms expressed through series, we find a derivation of the power series for the exponential function a^z starting from the formula

$$a^\omega = 1 + k\omega. \quad (1)$$

Here ω is infinitely small, while k is finite. Euler specifically describes the infinitesimal ω as being nonzero; see Sect. 4.8. Euler then raises Eq. (1) to the infinitely great power $i = \frac{z}{\omega}$ for a finite z to give

$$a^z = a^{i\omega} = (1 + k\omega)^i. \quad (2)$$

He then expands the right hand side of (2) into a power series by means of the binomial formula. In the chapters that follow, Euler finds infinite product decompositions for transcendental functions (see Sect. 3.5 below where we analyze his infinite product formula for sine). In this section, we argue that the underlying principles of Euler's mathematics are closer to Leibniz's than is generally recognized.

3.2 Useful Fictions

We argue in this subsection that Euler follows Leibniz both ontologically and methodologically. On the one hand, Euler embraces infinities as well-founded fictions; on the other, he distinguishes assignable quantities from inassignable quantities.

The nature of infinitesimal and infinitely large quantities is dealt with in Chapter 3 of *Institutiones Calculi Differentialis* (Euler 1755). We cite Blanton's English translation of the Latin original:

[e]ven if someone denies that infinite numbers really exist in this world, still in mathematical speculations there arise questions to which answers cannot be given unless we admit an infinite number.⁴ (Euler 2000, §82)

⁴ In the original Latin this reads as follows: "Verum ut ad propositum revertamus, etiamsi quis neget in mundo numerum infinitum revera existere; tamen in speculationibus mathematicis saepissime occurrunt questiones, ad quas, nisi numerus infinitus admittatur, responderi non posset." Note that the Latin uses the subjunctive *neget* (rather than *negat*), which is the mode used for a "future less vivid" condition: not "even if someone denies" but rather "even if someone were to deny."

Here Euler argues that infinite numbers are necessary “in mathematical speculations” even if someone were to deny “their existence in this world”. Does this passage indicate that Euler countenances the possibility of *denying* that “infinite numbers really exist in this world”? His position can be fruitfully compared with that of the scholars of the preceding generation. Those disagreed on the issue of the existence of infinitesimal quantities. Bernoulli, l’Hôpital, and Varignon staunchly adhered to the existence of infinitesimals, while Leibniz adopted a more nuanced stance. Leibniz’s correspondence emphasized two aspects of infinitesimal and infinite quantities: they are

- (1) *useful fictions* and
- (2) *inassignable quantities*.

It is important to clarify the meaning of the Leibnizian term *fiction*. Infinitesimals are to be understood as *pure fictions* rather than *logical fictions*, as discussed in Sect. 1; see Katz and Sherry (2012), Katz and Sherry (2013), and Sherry and Katz (2014). Furthermore, the work Jessep (2015) shows that Leibniz’s strategy for paraphrasing B-methods in terms of A-methods has to presume the correctness of an infinitesimal inference (more precisely, an inference exploiting infinitesimals), namely identifying the tangent to a curve. In the case of conic sections this succeeds because the tangents are already known from Apollonius. But for general curves, and in particular for transcendental curves treated by Leibniz,⁵ non-Archimedean infinitesimals remain an irreducible part of the Leibnizian framework, contrary to Ishiguro (1990, chapter 5). This argument is developed in more detail in Bascelli et al. (2016).

Similarly to Leibniz, Euler exploited the dichotomy of assignable versus inassignable, and mentioned the definition of infinitesimals as being smaller than every assignable quantity, as well as the definition of infinite numbers as being greater than every assignable quantity; see Gordon et al. (2002, 17, 19f.). Thus, Euler writes: “if z becomes a quantity less than any assignable quantity, that is, infinitely small, then it is necessary that the value of the fraction $1/z$ becomes greater than any assignable quantity and hence infinite” (Euler 2000, §90).

Euler’s wording in (Euler 2000, §82), making the usefulness of infinite numbers *independent* of their “existence in this world,” suggests that his position is closer to a Leibnizian view that infinitesimals are useful (or well-founded) fictions. Euler goes on to note that

an infinitely small quantity is nothing but a vanishing quantity, and so it is really = 0.⁶ (Euler 2000, §83)

Euler’s term *nihil* is usually translated as *nothing* by Blanton. However, in *Introductio*, §114, Blanton translates “tantum non nihilo sit aequalis” as “just not equal to zero” where it should be “just not equal to nothing” (see Sect. 4.8). Granted, “equal to nothing” would sound awkward, but Euler seems to distinguish it from “equal to zero”. It is tempting to conjecture that *nihil* might be equivalent to “exactly equal to zero”, whereas *cyphra* is the term for a quantity whose only possible assignable value is zero, or “shadow zero”. Meanwhile in *Institutiones* §84, Euler writes “duae quaevis cyphrae ita inter se sunt

⁵ Leibniz applies his method in his *de Quadratura Arithmetica* to find the quadrature of general cycloidal segments (Edwards 1979, 251). Here also the calculation exploits the family of tangent lines.

⁶ In the original Latin this reads as follows: “Sed quantitas infinite parva nil aliud est nisi quantitas evanescens, ideoque revera erit = 0.” Note that the equality sign “=” and the digit “0” are both in the original. While Euler writes “revera erit = 0” in §83, in the next §84 the formulation is “revera esse cyphram.”

aequales, ut earum differentia fit nihil.” This can be translated as follows: “two zeros are equal to each other, so that there is no difference between them.” This phrase is part of a larger sentence that reads as follows in translation: “Although two zeros are equal to each other, so that there is no difference between them, nevertheless, since we have two ways to compare them, either arithmetic or geometric, let us look at quotients of quantities to be compared in order to see the difference” (Euler 2000, 51).

This could be interpreted as saying that two instances of *cyphra* could be equal arithmetically but not geometrically. The distinction between *cyphra* and *nihil* could potentially give a satisfactory account for the Eulerian hierarchy of zeros.

3.3 Law of Homogeneity from Leibniz to Euler

As analyzed in Sect. 3.2, Euler insists that the relation of equality holds between any infinitesimal and zero. Similarly, Leibniz worked with a generalized relation of “equality” which was an *equality up to* a negligible term. Leibniz codified this relation in terms of his *transcendental law of homogeneity* (TLH), or *lex homogeneorum transcendentalis* in the original Latin Leibniz (1710). Leibniz had already referred to the law of homogeneity in his first work on the calculus: “quantitates differentiales, quae solae supersunt, nempe dx , dy , semper reperiuntur extra nominatores et vincula, et unumquodque membrum afficitur vel per dx , vel per dy , servata semper *lege homogeneorum* quoad has duas quantitates, quomodocunque implicatus sit calculus” Leibniz (1684) (emphasis added). This can be translated as follows: “the only remaining differential quantities, namely dx , dy , are found always outside the numerators and roots, and each member is acted on by either dx , or by dy , always with the law of homogeneity maintained with regard to these two quantities, in whatever manner the calculation may turn out.”

The TLH governs equations involving differentials. Bos interprets it as follows:

A quantity which is infinitely small with respect to another quantity can be neglected if compared with that quantity. Thus all terms in an equation except those of the highest order of infinity, or the lowest order of infinite smallness, can be discarded. For instance,

$$\begin{aligned} a + dx &= a \\ dx + ddy &= dx \end{aligned} \tag{3}$$

etc. The resulting equations satisfy this ... requirement of homogeneity. (Bos 1974, 33)

(here the expression ddx denotes a second-order differential obtained as a second difference). Thus, formulas like Euler’s

$$a + dx = a \tag{4}$$

(where a “is any finite quantity”, Euler 2000, §§86, 87) belong in the Leibnizian tradition of drawing inferences in accordance with the TLH and as reported by Bos in formula (3) above. The principle of cancellation of infinitesimals was, of course, the very basis of the technique, as articulated for example in l’Hôpital (1696) (see also Sect. 4.1). However, it was also the target of Berkeley’s charge of a logical inconsistency (Berkeley 1734). This

can be expressed in modern notation by the conjunction $(dx \neq 0) \wedge (dx = 0)$. But the Leibnizian framework does not suffer from an inconsistency of type $(dx \neq 0) \wedge (dx = 0)$ given the more general relation of “equality up to”; in other words, the dx is not identical to zero but is merely *discarded* at the end of the calculation in accordance with the TLH; see further in Sect. 4.13.

3.4 Relations (*pl.*) of Equality

What Euler and Leibniz appear to have realized more clearly than their contemporaries is that there is more than one relation falling under the general heading of “equality”. Thus, to explain formulas like (4), Euler elaborated two distinct ways, arithmetic and geometric, of comparing quantities. He described the two modalities of comparison in the following terms:

Since we are going to show that an infinitely small quantity is really zero [*cyphra*], we must meet the objection of why we do not always use the same symbol 0 for infinitely small quantities, rather than some special ones... [S]ince we have two ways to compare them [a more precise translation would be “there are two modalities of comparison”], either *arithmetic* or *geometric*, let us look at the quotients of quantities to be compared in order to see the difference. (Euler 2000 §84)

Furthermore,

If we accept the notation used in the analysis of the infinite, then dx indicates a quantity that is infinitely small, so that both $dx = 0$ and $adx = 0$, where a is any finite quantity. Despite this, the *geometric* ratio $adx : dx$ is finite, namely $a : 1$. For this reason, these two infinitely small quantities, dx and adx , both being equal to 0, cannot be confused when we consider their ratio. In a similar way, we will deal with infinitely small quantities dx and dy . (*ibid.*, emphasis added)

Having defined the two modalities of comparison of quantities, arithmetic and geometric, Euler proceeds to clarify the difference between them as follows:

Let a be a finite quantity and let dx be infinitely small. The arithmetic ratio of equals is clear: Since $ndx = 0$, we have

$$a \pm ndx - a = 0. \quad (5)$$

On the other hand, the geometric ratio is clearly of equals, since

$$\frac{a \pm ndx}{a} = 1. \quad (6)$$

(Euler 2000, §87).

While Euler speaks of distinct modalities of comparison, he writes them down symbolically in terms of two *distinct* relations, both denoted by the equality sign “=”; namely, (5) and (6). Euler concludes as follows:

From this we obtain the well-known rule that *the infinitely small vanishes in comparison with the finite and hence can be neglected [with respect to it]*. (Euler 2000, §87, emphasis in the original)

The “well-known rule” is an allusion to l’Hôpital’s *Demande ou Supposition* discussed in Sect. 4.1.

Note that in the Latin original, the italicized phrase reads *infinite parva prae finitis evanescent, atque adeo horum respectu reiici queant*. The words “with respect to it” (*horum respectu*) do not appear in Blanton’s translation. We restored them because of their importance for understanding Euler’s phrase. The term *evanescent* can mean either *vanish* or *lapse*, but the term *prae* makes it read literally as “the infinitely small vanishes *before* (or *by the side of*) the finite”, implying that the infinitesimal disappears *because of* the finite, and only once it is *compared to* the finite.

To comment on Euler’s phrase in more detail, a possible interpretation is that any motion or activity involved in the term *evanescent* does not indicate that the infinitesimal quantity is a dynamic entity that is (in and of itself) in a state of disappearing, but rather is a *static* entity that changes, or disappears, only “with respect to” (*horum respectu*) a finite entity. To Euler, the infinitesimal has a different status depending on what it is being compared to. The passage suggests that Euler’s usage accords more closely with reasoning exploiting *static* infinitesimals than with dynamic limit-type reasoning.

Euler proceeds to present the usual rules going back to Leibniz, L’Hôpital, and the Bernoullis, such as

$$a dx^m + b dx^n = a dx^m \tag{7}$$

provided $m < n$ “since dx^n vanishes compared with dx^m ” (ibid., §89), relying on his geometric comparison. Euler introduces a distinction between infinitesimals of different order, and directly *computes* a ratio $\frac{dx \pm dx^2}{dx}$ of two particular infinitesimals by means of the calculation

$$\frac{dx \pm dx^2}{dx} = 1 \pm dx = 1, \tag{8}$$

assigning the value 1 to it (ibid., §88). Note that rather than proving that the expression is equal to 1 (such *indirect* proofs are a trademark of the ϵ, δ approach), Euler *directly* computes (what would today be formalized as the *standard part* of) the expression.⁷ Euler combines the informal and formal stages by discarding the higher-order infinitesimal as in (6) and (8). Such an inferential move is formalized in modern infinitesimal analysis in terms of the standard part function or *shadow*; see Sect. 4.2. Euler concludes:

Although all of them [infinitely small quantities] are equal to 0, still they must be carefully distinguished one from the other if we are to pay attention to their mutual relationships, which has been explained through a geometric ratio (ibid., §89).

Like Leibniz in his *Symbolismus* (Leibniz 1710), Euler considers more than one way of comparing quantities. Euler’s formula (6) indicates that his geometric comparison is procedurally identical with the Leibnizian TLH (see Sect. 3.3): namely, both Euler’s geometric comparison and Leibniz’s TLH involve discarding higher-order terms in the context of a generalized relation of equality, as in (6) and (7).

Note that there were alternative theories around 1700, such as the one proposed by Nieuwentijt. Nieuwentijt’s system, unlike Leibniz’s system, possessed only first-order

⁷ To give an elementary example, the determination of the limit $\lim_{x \rightarrow 0} \frac{x+x^2}{x}$ in the ϵ, δ approach would involve first *guessing* the correct answer, $L = 1$, by using informal reasoning with *small* quantities; and then formally choosing a suitable δ for every ϵ in such a way that $\frac{x+x^2}{x}$ turns out to be within ϵ of L if $|x| < \delta$.

infinitesimals with square zero (Nieuwentijt 1695; Vermij 1989; Mancosu 1996, chapter 6). It is clear that the Eulerian hierarchy of orders of infinitesimals follows Leibniz’s lead.

Euler’s geometric comparison was dubbed “the principle of cancellation” in Ferraro (2004, 47); see Sect. 4.4 for a more detailed discussion of Euler’s *zero infinitesimals*.

3.5 Infinite Product Formula for Sine

In Sect. 2.5 we analyzed a pair of approaches to interpreting the work of the pioneers of analysis, namely the A-track in the context of an Archimedean continuum, and the B-track in the context of a Bernoullian continuum (an infinitesimal-enriched continuum). We explore a B-track framework as a proxy for the Eulerian *procedures*; here we leave aside the *ontological* or foundational issues, as discussed in Sect. 2.4. We will analyze specific procedures and inferential moves in Euler’s oeuvre and argue that the essential use he makes of both infinitesimals and infinite integers is accounted for more successfully in a B-track framework.

The fruitfulness of Euler’s approach based on infinitesimals can be illustrated by some of the remarkable applications he obtained. Thus, Euler derived an infinite product decomposition for the sine and sinh functions of the following form:

$$\sinh x = x \left(1 + \frac{x^2}{\pi^2}\right) \left(1 + \frac{x^2}{4\pi^2}\right) \left(1 + \frac{x^2}{9\pi^2}\right) \left(1 + \frac{x^2}{16\pi^2}\right) \dots \tag{9}$$

$$\sin x = x \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \left(1 - \frac{x^2}{16\pi^2}\right) \dots \tag{10}$$

(see Euler 1748, §§155–164). Here (10) generalizes an infinite product formula for $\frac{\pi}{4}$ (or $\frac{\pi}{2}$) due to Wallis; see Wallis 1656/2004, Proposition 191. Namely, Wallis obtained the following infinite product:

$$\prod_{n=1} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \dots = \frac{\pi}{2}.$$

Evaluating Euler’s product decomposition $\frac{\sin x}{x} = \prod_{n=1} \left(1 - \frac{x^2}{n^2\pi^2}\right)$ at $x = \frac{\pi}{2}$ one obtains $\frac{2}{\pi} = \prod_{n=1} \left(1 - \frac{1}{4n^2}\right)$ or $\frac{\pi}{2} = \prod_{n=1} \left(\frac{4n^2}{4n^2-1}\right)$. It follows that $\frac{\pi}{2} = \prod_{n=1} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1}\right)$, in other words $\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \dots$.

Euler also summed the inverse square series: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$; this is the so-called *Basel problem*. This identity results from (10) by comparing the coefficient of x^3 of the two sides and using the Maclaurin series for sine. This is one of Euler’s four solutions to the Basel problem; see Sandifer (2007, 111).

A common feature of these formulas is that Euler’s computations involve not only infinitesimals but also infinitely large natural numbers, which Euler sometimes treats as if they were ordinary natural numbers.

Euler’s proof of the product decompositions (9) and (10) rely on infinitesimalist procedures that find close proxies in modern infinitesimal frameworks. In “Appendix” we present a detailed analysis of Euler’s proof.

3.6 Law of Continuity

Euler’s working assumption is that infinite numbers satisfy the same rules of arithmetic as ordinary numbers. Thus, he applies the binomial formula to the case of an infinite exponent i without any further ado in Euler (1748, §115); see formula (2) above. The assumption was given the following expression in 1755:

The analysis of the infinite, which we begin to treat now, is nothing but a special case of the method of differences, explained in the first chapter, wherein the differences are *infinitely small*, while previously the differences were assumed to be *finite*. (Euler 2000, §114, emphasis added)

The significance of this passage was realized by Bos (who gives a slightly different translation; see Sect. 4.10). Euler’s assumption is consonant with the Leibnizian *law of continuity*: *il se trouve que les règles du fini réussissent dans l’infini... et que vice versa les règles de l’infini réussissent dans le fini*” (Leibniz 1702), though apparently Euler does not refer explicitly to the latter in this particular sense. Robinson wrote:

Leibniz did say ... that what succeeds for the finite numbers succeeds also for the infinite numbers and vice versa, and this is remarkably close to our *transfer of statements* from \mathbb{R} to ${}^*\mathbb{R}$ and in the opposite direction. (Robinson 1966, 266)

On the *transfer principle* see Sect. 4.6. Euler treats infinite series as polynomials of a specific infinite order (see Sect. 4.9 for a discussion of the difference between finite and infinite sums in Euler). In the context of a discussion of the infinite product

$$\left(1 + \frac{x}{i} + \frac{x^2}{4\pi^2}\right) \left(1 + \frac{x}{i} + \frac{x^2}{16\pi^2}\right) \left(1 + \frac{x}{i} + \frac{x^2}{36\pi^2}\right) \left(1 + \frac{x}{i} + \frac{x^2}{64\pi^2}\right) \cdots, \tag{11}$$

where i is an infinite integer, Euler notes that a summand given by an infinitesimal fraction $\frac{x}{i}$ occurs in each factor. One may be tempted therefore to discard it. The reason such an infinitesimal summand cannot be discarded according to Euler, is because it affects infinitely many factors:

through the multiplication of all factors, which are $\frac{1}{2}i$ in number [i being an infinitely large integer], there is a produced term $\frac{x}{2}$, so that $\frac{x}{i}$ cannot be omitted. (Euler 1748, §156)

In more detail, when one has a single factor, one can typically neglect the infinitesimal $\frac{x}{i}$. However, in this case one has $\frac{i}{2}$ factors, and the linear term in the product will be the sum of the linear terms in each factor. This is one of the Vieta rules that still holds when i is infinite by the *law of continuity*. Altogether there are $\frac{i}{2}$ factors, each of which contains a linear term $\frac{x}{i}$. Therefore altogether one obtains a contribution of $\frac{i}{2} \cdot \frac{x}{i} = \frac{x}{2}$, which is appreciable (noninfinitesimal) and therefore cannot be neglected.

Euler’s comment in 1748 shows that he clearly realizes that the infinitesimal $\frac{x}{i}$ present in each of the factors of (11) cannot be discarded at will. While in 1755, the *preliminary* status of the infinitesimal is officially “zero”, in actual calculations Euler does not rely on such preliminary declarations, as noted by Bos (see Sects. 4.5, 4.10).

Leibniz's differentials dx were infinitesimals, and while Leibniz did also consider non-infinitesimal differentials, he always denoted them by the symbol $(d)x$ rather than dx ;⁸ see Sect. 3.2 for a discussion of Leibnizian infinitesimals. There does not seem to be a compelling reason to think that Euler's dx 's were not infinitesimals, either. Ferraro appears to acknowledge this point when he writes: "Euler often simply treats differentials and infinitesimals as the same thing (for instance, see Euler [1755, 70])" (Ferraro 2004, 35, note 2). Indeed, the formula $\omega = dx$ appears in (Euler 1755, §118).

Note that Euler explicitly refers to the *number* of factors in his infinite product, expressed by a specific infinite integer. Similarly, when he applies the binomial formula $(a + b)^i$ with an infinite exponent i , there is an implied *final* term, or *terminal* summand, such as b^i , though it never appears explicitly in the formulas (see Sect. 4.9). We will analyze Euler's proof in detail in "Appendix".

3.7 The Original Rule of l'Hôpital

Euler's use of l'Hôpital's rule needs to be understood in its historical context. Most calculus courses today present the so-called l'Hôpital's rule in a setting purged of infinitesimals. It is important to set the record straight as to the nature of the original rule as presented by l'Hôpital in his *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes*.

Two points should be kept in mind here. First, L'Hôpital did *not* formulate his rule in terms of accumulation points, limits, epsilons, and deltas, but rather in terms of infinitesimals:

Cela posé, si l'on imagine une appliquée bd *infiniment proche* de BD , & qui rencontre les lignes courbes ANB , COB aux points f , g ; l'on aura $bd = \frac{AB \times bf}{bg}$, laquelle* ne diffère pas de BD . (l'Hôpital 1696, 145, emphasis added)

A note in the right margin at the level of the asterisk following the word *laquelle* reads "Art. 1." The asterisk refers the reader to the following item:

I. Demande ou Supposition.

... On demande qu'on puisse prendre indifféremment l'une pour l'autre deux quantités qui ne different entr'elles que d'une quantité *infiniment petite*. (l'Hôpital 1696, 2, emphasis added)

Clearly, Euler relied on l'Hôpital's original version of the rule rather than any modern paraphrase thereof. The original version of l'Hôpital's rule exploited infinitesimals. It seems reasonable therefore that if one were to seek to *understand* Euler's procedures in a modern framework, it would be preferable to do so in a modern framework that features infinitesimals rather than in one that doesn't.

Our second point is that Euler's procedures admit a B-track interpretation in terms of an infinitesimal value of z , and a relation

$$\lambda \approx \frac{1 - x^z}{z} \quad (12)$$

of being infinitely close, or Euler's *geometric comparison*; see Sect. 3.4. These concepts are, on the one hand, closer to Euler's world, and, on the other, admit rigorous proxies in

⁸ Note that Bos (1974) used the notation $\underline{d}x$ for Leibniz's $(d)x$.

the context of a modern B-continuum (such as the hyperreals), namely the relation $\lambda = \text{st}\left(\frac{1-x^z}{z}\right)$ involving the standard part function “st”. Arguably, the B-track formula (12) is a better proxy for *understanding* Euler’s infinitesimal argument than is Ferraro’s A-track formula (13).

3.8 Euclid’s Quantity

The classical notion of quantity is Euclid’s *μέγεθος* (magnitude). The general term *magnitude* covers line segments, triangles, rectangles, squares, convex polygons, angles, arcs of circles and solids. A general theory of magnitude is developed in the *Elements*, Book V. In fact, Book V is a masterpiece of deductive development. By formalizing its definitions (see below the formalisation of Definition V.4) and the tacit assumptions behind its proofs, one can reconstruct Book V and its 25 propositions as an axiomatic theory. Beckmann (1967/1968) and Błaszczyk and Mrówka (2013, 101–122) provide detailed sources for the axioms below in the primary source (Euclid 2007). See also Mueller (1981, 118–148), who mostly follows Beckmann’s development. Heiberg (1883–1888) is the standard modern edition of *Elements*.

As a result, Euclid’s magnitudes of the same kind (line segments being of one kind, triangles being of another, etc.) can be formalized as an ordered additive semigroup with a total order $<$ characterized by the following five axioms:

- E1 $(\forall x, y)(\exists n \in \mathbb{N})[nx > y]$,
- E2 $(\forall x, y)(\exists z)[x < y \Rightarrow x + z = y]$,
- E3 $(\forall x, y, z)[x < y \Rightarrow x + z < y + z]$,
- E4 $(\forall x)(\forall n \in \mathbb{N})(\exists y)[x = ny]$,
- E5 $(\forall x, y, z)(\exists v)[x : y :: z : v]$.

Here axiom E1 formalizes *Elements*, Definition V.4. More specifically, Euclid’s definition reads:

Magnitudes [such as a, b] are said to have a ratio with respect to one another which, being multiplied [i.e., na] are capable of exceeding one another [i.e., $na > b$].

The definition can be formalized as follows: $(\forall a, b)(\exists n)(na > b)$. This reading of Euclid V.4 is a standard interpretation among historians; see Beckmann (1967/1968, 31–34), Mueller (1981, 139), De Risi (2016, section II.3).

The early modern mathematics developed largely without reference to the Archimedean axiom. Some medieval editions of *Elements* simply omitted the definition V.4; more precisely, they give *Proportion is a similarity of ratios* instead of definition V.4 of our modern editions; see Grant (1974, 137). The same applies to C. Clavius’ *Euclidis Elementorum*, one of the most popular seventeenth century edition of *Elements*; see (Clavius 1589, 529).

We do not find any explicit reference to the Archimedean axiom in the works of Stevin, Descartes, Newton though there is a mention of Euclid’s axiom in Leibniz’s letter to l’Hôpital (Leibniz 1695, 288), nor in the works of Euler. Even the classical constructions of the real numbers provided in 1872 by Heine, Cantor and Dedekind contain no explicit mention of the Archimedean axiom, as it was recognized as such only in 1880s by Stolz; see Sect. 3.9. The Archimedean axiom follows from the continuity axiom (Dedekind axiom) and is equivalent to both the absence of infinitesimals and the cofinality of the integers within the reals defined in those constructions. It took time for mathematicians to understand the precise relation between the continuity axiom and the Archimedean axiom. It was not until 1901 that Hölder proved that the continuity axiom (more precisely, Dedekind axiom) implies the Archimedean axiom; see Hölder (1901, 1996).

3.9 Stolz and Heiberg

Stolz (1885) rediscovered the Archimedean axiom for mathematicians, making it one of his axioms for magnitudes. The Archimedean axiom was studied earlier in Stolz (1883), while Stolz (1885) was a popular and widely read book. Stolz coined the term *Archimedean axiom*. As the source of this axiom he points out Archimedes' treatises *On the sphere and cylinder* and *The quadrature of the parabola*. As regards Euclid, Stolz refers to books X and XII. He does not seem to have noticed that definition 4 of book V is related to the axiom of Archimedes. Johan L. Heiberg in his comment on the Archimedean axiom (lemma) cites Euclid's definition V.4 and observes that "these are the same axioms" (Heiberg 1881, 11). Possibly as a result of his comment Euclid's definition V.4 is called the Archimedean axiom.

At the end of the nineteenth century, Euclid's theory of magnitude was revived by Stolz (1885), Weber (1895), and Hölder (1901). These authors developed axiomatic theories of magnitude. For a modern account of these theories see Błaszczyk (2013). Despite certain differences, they all accept axioms E1–E4 of Sect. 3.8 as a common characterisation of magnitude. Instead of E5, some authors tend to use the Dedekind axiom of continuity, which implies E5. Hölder was the first one to show that E1 follows from E2–E4 and the Dedekind continuity axiom.

Thus, while axiom E1 is a feature of the classic and modern notion of magnitude, it is absent from Euler's characterisation of quantity. Moreover, Euler is explicit about the existence of infinite quantities; see Sect. 3.10.

In the Eulerian context, a magnitude, or quantity, is not (yet) a number. Euler's quantities are converted to numbers once one specifies an arbitrary quantity as the *unit*, or *unity*. In addition to a unity, Euler needs a notion of a ratio. Euler's definition is similar to Newton's:

the determination, or the measure of magnitude of all kinds, is reduced to this: fix at pleasure upon any one known magnitude of the same species with that which is to be determined, and consider it as the *measure* or *unit*; then, determine the proportion [ratio] of the proposed magnitude to this known measure. This proportion [ratio] is always expressed by numbers; so that a number is nothing but the proportion [ratio] of one magnitude to another arbitrarily assumed as the unit. (Euler 1771, §4)

However, neither Newton nor Euler provided a definition of *ratio*. The term *proportion* corresponds to the term *Verhältnis* (ratio) in the German edition of Euler's *Algebra*, and to *rapport* (ratio) in the French edition (Euler 1807).

3.10 Euler on Infinite Numbers and Quantities

Euler is explicit about the existence of infinite (and therefore non-Archimedean) quantities and numbers:

not only is it possible to give a quantity of this kind, to which increments are added without limit, a certain character, and with due care to introduce it into calculus, as we shall soon see at length, but also there exist real cases, at least they can be conceived, in which an infinite number actually exists. (Euler 2000, §75)

Euler's important qualification "at least they can be conceived" with regard to the existence of infinite numbers is consistent with the Leibnizian idea of them as *useful fictions*; see Sect. 3.3.

4 Critique of Ferraro's Approach

Ferraro's recent text on Euler seeks to steer clear of certain interpretive approaches to Euler:

“My point of view is different from that of some recent papers, such as McKinzie-Tuckey (1997) and Pourciau (2001). In this writing the authors recast the early procedures directly in terms of the modern foundation of analysis or interpret the earlier results in terms of modern theory of non-standard analysis and understand the results in the light of this later context.” (Ferraro 2012, 2).

Ferraro's 2012 piece has significant textual overlap with his article from 2004. Here Ferraro asserts that “one can see in operation in their writings a conception of mathematics which is quite *extraneous* to that of Euler ... the attempt to specify Euler's notions by applying modern concepts is only possible if elements are used which are essentially *alien* to them, and thus Eulerian mathematics is transformed into something wholly different” (Ferraro 2004, 51f., emphasis added). In 2004 Ferraro included two articles by Laugwitz in the list of such allegedly “extraneous” and “alien” approaches: the article Laugwitz (1989) in *Archive for History of Exact Sciences*, as well as Laugwitz (1992).

Ferraro's comments here betray an insufficient sensitivity to the distinction analyzed in Sect. 2.4, namely, isolating methodological concerns from obvious problems of ontology as far as Euler's infinitesimals are concerned. Granted, modern set-theoretic frameworks, customarily taken to be an *ontological* account of the foundations of mathematics, are alien to Euler's world. But is Laugwitz's approach to Euler's *methodology* really “extraneous” or “alien” to Euler? Interpretive approaches seek to clarify Euler's mathematical *procedures* through the lens of modern formalisations. In the passage cited above, Ferraro appears initially to reject such approaches, whether they rely on modern ϵ, δ interpretations à la Weierstrass, or on infinitesimal interpretations à la Robinson. Yet in 2004, Ferraro writes:

I am not claiming that 18th-century mathematics should be investigated without considering modern theories. Modern concepts are essential for *understanding* eighteenth century notions and why these led to *meaningful* results, even when certain procedures, puzzling from the present views, were used. (Ferraro 2004, 52, emphasis added)

Thus, in the end Ferraro does need modern theories to “understand” (as he puts it) Euler, even though such procedures are “meaningless” to the latter. Ferraro's position needs to be clarified, since any modern attempt to *understand* Euler will necessarily *interpret* him, as well. While rejecting Laugwitz's interpretive approach to Euler, Ferraro does seek to *understand*, and therefore *interpret*, Euler by modern means. To pinpoint the difference between Laugwitz's interpretive approach (rejected by Ferraro) and Ferraro's own interpretive approach, let us examine a sample of Ferraro's reading of Euler.

4.1 From l'Hôpital and Euler to Epsilon and Delta

Ferraro deals with an infinitesimal calculation in (Euler 1730–1731, 11–12) where Euler sought the value of

$$\frac{1 - x^{\frac{g}{j+g}}}{g}$$

for $f = 1$ and $g = 0$ by applying l'Hôpital's rule to $\frac{1-x^c}{z}$. Ferraro proceeds to present the problem "from a modern perspective" by analyzing the function $f(z) = \frac{1-x^z}{z}$ and its behavior near $z = 0$ in the passage already cited in Sect. 2.1, featuring the formula

$$\lambda = \lim_{z \rightarrow c} f(z) \tag{13}$$

(as already noted, c must be replaced by 0). Here the formula label (13) is added for later reference.

On the face of it, Ferraro merely explains what it means to a modern reader to extend a function by continuity at a point where the function is undefined. However, Ferraro's presentation of a modern explanation, with its talk of accumulation points, limits, epsilons, and deltas, is firmly grounded in an A-track interpretation of the Eulerian calculation using l'Hôpital's rule. But why should one seek to "understand" Euler using A-methodology?

In Sect. 3.7 we placed Euler's use of l'Hôpital's rule in its historical context. What Ferraro presents here is an (ϵ, δ) à la Weierstrass formalisation of Euler's procedure. He goes on to point out that such an approach would be "meaningless" to Euler. Nevertheless, Ferraro goes on to make the following remarkable claim:

there is something in common between the Eulerian procedure and the modern one based upon the notion of limit: evanescent quantities and endlessly increasing quantities were based upon an intuitive and primordial idea of two quantities approaching each other. I refer to this idea as "protolimit" to avoid any possibility of a modern interpretation. (Ferraro 2004, 46)

Thus according to Ferraro, there is "something in common between the Eulerian procedure and the modern one", after all. Ferraro's *protolimit* is intended to be different from the (A-track) limit. But shouldn't we rather interpret Eulerian infinitesimals in terms of, say, a *protoshadow*? The term *shadow* is sometimes employed to refer to the (B-track) *standard part function*, discussed in Sect. 4.2.

4.2 Shadow

In any totally ordered field extension E of \mathbb{R} , every finite element $x \in E$ is infinitely close to a suitable unique element, namely its standard part $x_0 \in \mathbb{R}$.⁹

Ferraro finds fault with the standard part function as a tool in interpreting Euler's equality $\frac{i-1}{i} = 1$. Ferraro writes that the equality "should not be intended as $\frac{i-1}{i} \approx 1$ " (Ferraro 2004, 49) and provides the following clarification in footnote 36 on page 49: "By $a \approx b$, I mean that the difference $a - b$ is an infinitesimal hyperreal number." The criticism recurs in (Ferraro 2012, 10) where the standard part function is mentioned explicitly. However, this criticism only raises an issue if one assumes that Euler's equalities were not approximate but rather exact equalities. Such an assumption may be too simplistic a reading of Euler's stance on *arithmetic and geometric comparisons*; see Sect. 3. See also "Appendix", Step 5 and formula (28) where Euler wrote that the term x^2/i^2 is negligible in each of the factors of $e^x - e^{-x}$ *only because* the number of the said factors is small compared to i^2 .

⁹ Indeed, via the total order, the element x defines a Dedekind cut on \mathbb{R} . By the usual procedure, the cut specifies a real number $x_0 \in \mathbb{R} \subseteq E$. The number x_0 is infinitely close to $x \in E$. The subring $E_f \subseteq E$ consisting of the finite (i.e., *limited*) elements of E therefore admits a map $st : E_f \rightarrow \mathbb{R}$, $x \mapsto x_0$, called the *standard part function*, or *shadow*, whose role is to *round off* each finite (limited) x to the nearest real x_0 .

Ferraro wrote that “from the modern perspective, the problem of extending the function” is interpreted in terms of accumulation points, A-track limits, epsilons, and deltas.

But couldn't we perhaps surmise, instead, that “from a modern perspective, the problem of extending the function may involve infinitesimals, the relation of being infinitely close, and standard part”?

Ferraro's claim that Eulerian infinitesimals “were symbols that represented a primordial and intuitive idea of limit” (Ferraro 2004, 34), with its exclusive focus on the limit concept in its generic meaning, tends to blur the distinction between the rival Weierstrassian and modern-infinitesimal methodologies (see Sect. 2.5). Eulerian infinitesimals are intrinsically not Archimedean but rather follow the methodology of his teacher Bernoulli, co-founder with Leibniz of what we refer to as the B-track. A better methodological proxy for Eulerian infinitesimals than Ferraro's “primordial limit” is provided by a modern B-track approach to analysis, fundamentally different from Ferraro's A-track (proto)limit.

Meanwhile, Laugwitz sought to formalize Euler's procedures in terms of modern infinitesimal methodologies. It emerges that, while Ferraro's own A-track reading is deemed “essential for understanding eighteenth-century notions and why these led to meaningful results” as claimed in Ferraro (2012, 2), Laugwitz's infinitesimal interpretation is rejected by Ferraro as being both “extraneous” and “alien” to Euler's mathematics. In short, Laugwitz's interpretation does not fit Ferraro's Procrustean A-track way of, as he put it, “understanding” Euler (Ferraro 2012, 2).

4.3 B-track Reading in Felix Klein

Laugwitz's interpretation accords with Felix Klein's remarks on the dual tracks for the development of analysis as found in Klein (1932, 214). In 1908, Felix Klein described a rivalry of the dual approaches as we saw in Sect. 2.5. Klein went on to formulate a criterion for what would qualify as a successful theory of infinitesimals. A similar criterion was formulated in Fraenkel (1928, 116f.). For a discussion of the Klein–Fraenkel criterion see (Kanovei et al. 2013, section 6.1). The criterion was formulated in terms of the mean value theorem. Klein concluded:

I will not say that progress in this direction is impossible, but it is true that none of the investigators have achieved anything positive. (Klein 1932, 219)

Thus, the B-track approach based on notions of infinitesimals is not limited to “the work of Fermat, Newton, Leibniz and many others in the seventeenth and eighteenth centuries”, as implied by Katz (2014). Rather, it was very much a current research topic in Felix Klein's mind. See Ehrlich (2006) for detailed coverage of the work on infinitesimals around 1900.

Of course, Klein had no idea at all of Robinson's hyperreal framework as first developed in Robinson (1961). What Klein was referring to is the *procedural* issue of how analysis is to be presented, rather than the *ontological* issue of a specific realisation of an infinitesimal-enriched field in the context of a traditional set theory; see Sect. 2.4.

Finally, we note that A-track readings of Euler tend to be *external* to Euler's *procedures*, whereas infinitesimal readings are *internal*,¹⁰ in the sense that it provides proxies for *both* the procedures and the results of the historical infinitesimal mathematics. This is possible because modern infinitesimal procedures incorporate both infinitesimals and infinite numbers as do Eulerian procedures. Meanwhile, the Weierstrassian approach tends

¹⁰ This use of the term *internal* is not to be confused with its technical meaning in the context of enlargements of superstructures; see Goldblatt (1998).

to provide proxies for the results but not the procedures, since both infinitesimals and infinite numbers have been eliminated in this approach.

4.4 Hidden Lemmas and Principle of Cancellation

Laugwitz argued that Euler’s derivation of the power series expansion of a^x contains a *hidden lemma*, to the effect that a certain *infinite sum* of infinitesimals is itself infinitesimal under suitable conditions; see Laugwitz (1989, 210). Namely, let i be infinite. Consider Euler’s formula

$$\left(1 + \frac{kz}{i}\right)^i = 1 + kz + \frac{i-1}{2i}k^2z^2 + \frac{(i-1)(i-2)}{2i \cdot 3i}k^3z^3 + \dots,$$

or alternatively

$$\left(1 + \frac{kz}{i}\right)^i = 1 + \frac{i}{i}kz + \frac{i(i-1)}{i \cdot 2i}k^2z^2 + \frac{i(i-1)(i-2)}{i \cdot 2i \cdot 3i}k^3z^3 + \dots \tag{14}$$

There are infinitely many summands on the right. (Euler 1748, §115–116) goes on to make the substitutions

$$\frac{i-1}{i} = 1, \frac{i-1}{2i} = \frac{1}{2}, \frac{i-2}{3i} = \frac{1}{3}, \dots \tag{15}$$

which he justifies by invoking the fact that i is infinite. The result is the exponential series. The effect of these changes is cumulative, since the products involved contain an ever increasing number of factors. Thus, one needs to make the substitution

$$(i-1)(i-2) \dots (i-n) = i^n$$

for each finite n in the righthand side, but there are still infinitely many summands affected. Each of these substitutions entails an infinitesimal change but there are infinitely many substitutions involved in evaluating (14).

Ferraro takes issue with Laugwitz’s contention in the following terms:

It is evident that Laugwitz’s remark arises from the interpretation of $\frac{i-1}{i} = 1$ as $\frac{i-1}{i} \approx 1$. This interpretation contrasts with the Eulerian statement that $a + dx = a$ is an exact equality and not an approximate one. (Ferraro 2004, 49)

Ferraro goes on to assert that, contrary to Laugwitz’s claim, Euler

did not see gaps in the proof of [the series expansion of a^x], and this was due to the fact that he understood $\frac{i-1}{i} = 1$ as a formal equality involving fictitious entities. (Ferraro 2004, 50)

Indeed, if $\frac{i-1}{i} = 1$ were an exact equality along with the other expressions in (15), the evaluation of the righthand side of (14) to the exponential series would be immediate and free of any gaps, as Ferraro contends.

Alas, Ferraro underestimates Euler’s perceptiveness here. Ferraro does not explain how an invocation of “a formal equality involving fictitious entities” deflects Laugwitz’s contention that Euler’s proof contains a *hidden lemma*. Ferraro’s insistence on the “exact equality” $\frac{i-1}{i} = 1$ suggests that the infinitesimal “error” in $\frac{i-1}{i} = 1$, or

$$1 - \frac{1}{i} = 1, \quad (16)$$

is to be understood as exactly zero. Declaring the infinitesimal “error” $\frac{1}{i}$ to be exactly zero would obviate the need for justifying the hidden lemma, since an infinite sum of zeros is still zero, or at any rate so Ferraro appears to interpret Euler’s argument. We will return to Ferraro’s “fictitious entities” in Sect. 4.8.

4.5 Two Problems with Ferraro’s Reading

There are two problems with Ferraro’s claim that Euler is invoking an exact equality with no infinitesimal error. First, Euler explicitly writes otherwise (see Sect. 3.6 on the issue of disappearing infinitesimals), and in fact in the calculation under discussion, Euler exploits the relation $z = \omega i$ (Euler 1748, §115) with infinitesimal ω and finite z , which would be quite impossible if ω were literally zero.

The second problem is that, as Ferraro himself noted in his recent text, Euler expressed the integral as “the sum of an infinite number of infinitesimals” (Ferraro 2012, 10). Euler expresses the integral in terms of the expression

$$\alpha(A + A' + A'' + A''' + \dots + X) \quad (17)$$

in (Euler 1768–1770, chapter VII, 184), where α is an infinitesimal step of a suitable partition, while A, A', A'', A''', \dots are the (finite) values of the integrand at (infinitely many) partition points. The quantities

$$\alpha A, \alpha A', \alpha A'', \alpha A''', \dots$$

are still infinitesimal, and therefore would be exactly zero, so that their infinite sum (17) would be paradoxically zero as well. Thus, such a reading of Euler’s reasoning attributes to him an alarming paradox not dealt with in Ferraro’s approach.

Ferraro mentions Euler’s interpretation of the integral as an infinite sum of infinitesimals in Ferraro (2004, 50, footnote 39), but fails to explain how the paradox mentioned in the previous paragraph could be resolved (other than implying that infinitesimals are sometimes zero, and sometimes not).

In sum, we agree with Bos’ evaluation of Euler’s preliminary remarks on “infinitesimals as zeros” as being at variance with his actual mathematical practice (see Sect. 4.10). It is unlikely that a literal interpretation of Euler’s preliminary remarks (that the infinitesimal is exactly zero) could give a fruitful way of interpreting Euler’s mathematics. Ferraro’s rejection of Laugwitz’s analysis of Euler’s argument in terms of a *hidden lemma* (requiring further justification) is therefore untenable.

4.6 Generality of Algebra

It was known already to Cauchy that some of Euler’s doctrines are unsatisfactory. More specifically, Cauchy was critical of Euler’s and Lagrange’s *generality of algebra*, to the effect that certain relations involving variable quantities are viewed as being valid even though they can fail for certain specific values of the variables. By the time mathematicians started analyzing Fourier series in the 1820s it became clear that some applications of the *generality of algebra* are untenable. Cauchy specifically rejects this principle in the introduction to his *Résumé des Leçons* (Cauchy 1823).

In the context of a discussion of Euler's principle of the *generality of algebra*, Ferraro notes that the Eulerian "general quantity"

was represented by graphic signs which were manipulated according to appropriate rules, which were the same rules that governed geometrical quantities or true numbers. (Ferraro 2004, 43)

The idea that "the same rules" should govern ideal/fictional numbers and "true numbers" is consonant with the Leibnizian *law of continuity*. The latter is arguably a fruitful methodological principle. It was formalized as the *transfer principle* in Robinson's framework.¹¹

Meanwhile, Ferraro fails to distinguish between, on the one hand, a historically fruitful *law of continuity*, and, on the other, the *generality of algebra* that was found to be lacking in the 19th century, as he continues in the next sentence:

The principle of the generality of algebra held: the rules were applied in general, regardless of their conditions of validity and the specific values of quantity. (ibid.)

Cauchy's critique of Euler's principle of the *generality of algebra* is well known to historians; it is an uncontroversial statement that certain elements of Euler's oeuvre need to be reinterpreted if one is to develop a consistent interpretation thereof. Another such element is the zero infinitesimal, as discussed in Sect. 4.7.

4.7 Unsettling Identity

The claim that the infinitesimal is exactly *equal* to zero occasionally does appear in Euler's writing, such as in the *Institutiones* in reference to dx . On the other hand, Euler specifically discusses varieties of the notion of *equality*, with the *geometric* notion being similar to the generalized relation of equality implied in the Leibnizian *transcendental law of homogeneity* (see Sect. 3.4). Even though at times Euler insists that his equality is *exact equality*, at other times he does envision more general modalities of comparison. Ferraro himself implicitly acknowledges this when he describes Euler's equality $a + dx = a$ as a "principle of cancellation" (Ferraro 2004, 47). The term *principle of cancellation* would appear to imply that there is something to *cancel*: not merely an *exact zero*, but a nonzero infinitesimal dx .

On an even more basic level, if for infinite i one has $\frac{1}{i} = 0$ as in (16), then multiplying out by i we obtain $1 = 0 \times i$, but 0 times any number is still 0, so that we would obtain an unsettling identity $1 = 0$ (at least if we interpret "=" as literal equality), in addition to the paradox with the integral mentioned above.

Similarly, Euler seeks to divide by an infinitesimal dx so as to obtain the differential ratio $\frac{dy}{dx}$. It follows that dx cannot be an exact zero if one is to have any hope for a consistent account of Euler's procedures. A notion of *zero infinitesimal* interpreted literally is arguably as problematic as some aspects of the principle of the *generality of algebra*

¹¹ The *transfer principle* is a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system, still apply (i.e., are "transferred") to an extended number system. Thus, the familiar extension $\mathbb{Q} \subseteq \mathbb{R}$ preserves the properties of an ordered field. To give a negative example, the extension $\mathbb{R} \subseteq \mathbb{R} \cup \{\pm\infty\}$ of the real numbers to the so-called *extended reals* does not preserve the properties of an ordered field. The hyperreal extension $\mathbb{R} \subseteq {}^*\mathbb{R}$ preserves *all* first-order properties, such as the identity $\sin^2 x + \cos^2 x = 1$ (valid for all hyperreal x , including infinitesimal and infinite values of $x \in {}^*\mathbb{R}$). For a more detailed discussion, see Keisler (1986).

already found to be lacking by Cauchy (see Sect. 4.6). It can be reinterpreted in terms of the distinction between *cyphra* and *nihil* as discussed in Sect. 3.2.

4.8 Fictitious Entities

Ferraro described the Eulerian substitution $\frac{i-1}{i} = 1$ as a “formal equality involving fictitious entities” in his text (Ferraro 2004, 50). It is not entirely clear how such an evocation of fictionality resolves the delicate mathematical problem posed by this substitution.

Two scientific generations earlier, Leibniz described infinitesimals as “useful fictions,” yet he did not think that just because infinitesimals are “fictions” one is allowed to set them equal to zero at will.

Ferraro further claimed that “The use of fictions made Eulerian mathematics extremely different from modern mathematics” in his text (Ferraro 2007, 64). The plausibility of the claim depends on equivocation on procedure/ontology as discussed in Sect. 2.4. The fact that what Ferraro has in mind here is ontology is made clear on the previous page where he writes: “The rules that Euler uses upon [*sic*] infinite and infinitesimal quantities constitute an immediate extrapolation of the behaviour of a finite variable i tending to ∞ or 0”, (Ferraro 2007, 63) and adds: “what is wholly missing is the *complex construction* of ${}^*\mathbb{R}$ and the assumptions upon which it is based” (*ibid.*, emphasis added.). But this ontological complaint is utterly irrelevant to procedure.

The infinitesimals in Leibniz and Euler may have been *fictions*. However, they were not *fictive* or purely rhetorical, as Ferraro appears to imply. Rather, they pose subtle issues of interpretation that are not resolved by an appeal to “formal equality involving fictitious entities”; see further in Sect. 4.12.

On occasion, Euler specifically wrote that his infinitesimals are *unequal* to zero: “Let ω be a number infinitely small, or a fraction so tiny that it is just not equal to zero (*tantum non nihilo sit aequalis*)” (Euler 1988, §114). This passage refers specifically to the infinitesimal ω in formula (1) used in the derivation of the power series of the exponential function (see Sect. 3.1 on the exponential function), showing that error estimates are indeed required even if one takes Euler literally.

4.9 Finite, Infinite, and Hyperfinite Sums

Euler’s use of infinite integers and their associated infinite products (such as the product decomposition of the sine function) were interpreted in Robinson’s framework in terms of hyperfinite expressions. Thus, Euler’s product of i -infinitely many factors in (10) is interpreted as a hyperfinite product in Kanovei and Reeken (2004, 74). A hyperfinite formalisation of Euler’s argument involving infinite integers and their associated products illustrates the successful formalisation of the *arguments* (and not merely the *results*) of classical infinitesimal mathematics.

In a footnote on eighteenth century notation, Ferraro presents a novel claim that “for eighteenth century mathematicians, there was no difference between finite and infinite sums” (Ferraro 1998, 294, footnote 8). Far from being a side comment, the claim is emphasized a decade later in the preface to his book: “a distinction between finite and infinite sums was lacking, and this gave rise to formal procedures consisting of the infinite extension of finite procedures” (Ferraro 2008, viii). The clue to decoding Ferraro’s claim is found in the same footnote, where Ferraro distinguishes between sums featuring a final term after the ellipsis, such as

$$a_1 + a_2 + \dots + a_n, \tag{18}$$

and “infinite sums” without such a final term, as in

$$a_1 + a_2 + \dots + a_n + \dots \tag{19}$$

Note that A-track syntax is unable to account for *terminating* infinite expressions which routinely occur in Euler. To be sure, Euler does *not* use his infinite i as a final index in infinite sums of type $a_1 + \dots + a_i$ common in modern infinitesimal frameworks. However, his binomial expansions with exponent i play the same role as the modern infinite sums $a_1 + \dots + a_i$. The final term a_i is hinted upon by means of Euler’s notation “&c.” but does not appear explicitly. Nonetheless, procedurally speaking his infinite sums play the same role as the modern $a_1 + \dots + a_i$.

From an A-track viewpoint, a terminating sum (18) is necessarily a finite one, whereas only expressions of the form (19) ending with an ellipsis allow for a possibility of an “infinite sum.” No other option is available in the A-track; yet Euler appears recklessly to write down *infinite* terminating expressions, as in the proof of the product formula for sine (see Sect. 3.6 for a discussion of terminal summands in infinite sums in Euler).

Meanwhile, the B-track approach allows one to account both for Euler’s *infinite* integer i and for *terminating* expressions containing i terms (see “Appendix” for an instance of Euler’s use of polynomials of infinite degree). Euler discusses the difference between finite and infinite sums in *Introductio* (Euler 1748, §59). Terminating *infinite* sums are easily formalized in Robinson’s framework in terms of hyperfinite expressions (see “Appendix”).

In a subsequent article, Ferraro and Panza write: “Power series were conceived of as quasi-polynomial entities (that is, mere *infinitary* extensions of polynomials)” (Ferraro and Panza 2003, 20, emphasis added), but don’t mention the fact that such an extension can be formalized in terms of hyperfinite expressions, perhaps out of concern that this may be deemed “alien” or “extraneous” to Euler.

Euler’s formula $a^{i\omega} = (1 + k\omega)^i$ is analyzed in Ferraro (2004, 48). Ferraro reformulates Euler’s formula in terms of modern *Sigma* notation as follows:

$$a^{i\omega} = (1 + k\omega)^i = \sum_{r=0}^{\infty} \binom{i}{r} (k\omega)^r.$$

The formula

$$a^x = \sum_{r=0}^{\infty} \frac{1}{r!} (kx)^r \tag{20}$$

appears in (Ferraro 2004, 48) and is attributed to Euler. The Sigma notation $\sum_{r=0}^{\infty}$ appears several times in Ferraro’s analysis and is clearly not a misprint; it appears again in (Ferraro 2007, 48, 54). Similarly, Ferraro exploits the modern notation $\sum_{i=1}^{\infty} a_i$ for the sum of the series, in Ferraro (2008, 5), while discussing late 16th (!) century texts of Viète. The Sigma notation (20) is familiar modern notation for infinite sums defined via the modern concept of limit in a Weierstrassian context. Note that formula (20) attributed by Ferraro to Euler involves assigning a sum to the series, namely a^x , and therefore is not merely a formal power series. The summation of an infinite series via the concept of limit (namely, limit of the sequence of partial sums) is not accessory but rather a *sine qua non* aspect of such summation (alternatively, one could take the standard part of a hyperfinite sum, but such an

approach is apparently not pursued by Ferraro). The symbol ∞ appears in (20) as a kind of subjunctive. It has no meaning other than a reminder that a limit was taken in the definition of the series. In modern notation, the symbol ∞ does not stand for an infinite integer (contrary to the original use of this symbol by Wallis).

Thus, Ferraro reformulates Euler's calculation using the Sigma notation for infinite sum, including the modern somewhat subjunctive use of the superscript ∞ . However, such a procedure is extraneous to Euler's mathematics, since Euler specifically denotes the (infinite) power by i . Applying the binomial formula with exponent i , one would obtain, not Ferraro's (20), but rather a sum from 0 to i , namely

$$\sum_{r=0}^i \binom{i}{r} (kx)^r. \quad (21)$$

Euler's derivation of the exponential series is analyzed in more detail in Sect. 4.4. Infinite sums of type (21) are perfectly meaningful when interpreted in Robinson's framework (see "Appendix"). Ferraro's anachronistic rewriting of Euler's formula betrays a lack of sensitivity to the actual mathematical content of Euler's work.

4.10 Bos–Ferraro Differences

In this section we compare Ferraro's take on Euler with the approaches by other scholars, more compatible with our reading of Euler. We will first compare the approaches of Bos and Ferraro to Euler scholarship, and then those of Ferraro and Laugwitz. Bos summarized Euler's preliminary discussion of infinitesimals in the following terms:

Euler claimed that infinitely small quantities are equal to zero, but that two quantities, both equal to zero, can have a determined ratio. This ratio of zeros was the real subject-matter of the differential calculus. (Bos 1974, 66)

Bos goes on to note that Euler's preliminary discussion is at variance with Euler's actual mathematical practices even in the *Institutiones* (and not merely in the *Introductio* as discussed in Sect. 3.6), where the properties of the infinitely small are similar to those of finite differences:

After having treated, in the first two chapters, the theory of finite difference sequences, he defined the differential calculus as the calculus of infinitesimal differences:

The analysis of infinites, with which I am dealing now, will be nothing else than a special case of the method of differences expounded in the first chapter, which occurs, when the differences, which previously were supposed finite, are taken infinitely small. (Euler 1755, §114)

which is rather at variance with his remarks quoted above, a *contradiction* which shows that his arguments about the infinitely small did not really influence his presentation of the calculus (Bos 1974, 67f., emphasis added).

Before analyzing Ferraro's reaction to this position, we note that Bos' focus on Euler's "presentation of the calculus" indicates a concern for methodological issues related to the nature of Euler's *procedures*, rather than focusing on the *ontological* nature of the objects (the infinitely small) that Euler utilizes, in line with the distinction between procedure and ontology that we explored in Sect. 2.4.

Ferraro disagrees with Bos' perception of a "contradiction" in Euler's writing:

According to Bos, there is "a contradiction which shows that his arguments about the infinitely small did not really influence his presentation of calculus" [Bos, 1974, 68–69]. However, I would argue that one may see a contradiction in the *Institutiones* only if, in contrast to Euler, [1] one distinguishes between *limits* and *infinitesimals* and [2] neglects the nature of evanescent quantities as fictions, [3] the role of formal manipulations and [4] the absence of a separation between semantics and syntax in the Eulerian calculus. (Ferraro 2004, 54, emphasis and numerals [1], [2], [3], [4] added)

Ferraro appears to suggest that Bos' position is problematic with regard to the four items enumerated above. We will not analyze all four, but note merely that in his item

[1] "one distinguishes between limits and infinitesimals", (22)

Ferraro commits himself explicitly to the position that "distinguish[ing] between limits and infinitesimals," as Bos does, is an inappropriate approach to interpreting Euler. Rather, Ferraro sees a conceptual continuity between limits and infinitesimals in Euler, or more precisely what he refers to as a "continuous leap" (see Sect. 4.11).

We argue that Bos's position on this aspect of Euler's oeuvre is more convincing than Ferraro's. Note that Euler's insistence on the similarity of the properties of the finite and infinitesimal differences, in the passage cited by Bos, is consonant with a Leibnizian *law of continuity*, which requires two types of quantities to be compared: assignable and inassignable (e.g., infinitesimal); see Sect. 3.3.

4.11 Was Euler Ambiguous or Confused?

Ferraro postulated a conceptual continuity between limits and infinitesimals in Euler's work, as expressed in Ferraro's comment (22) meant to be critical of Bos' position.

Ferraro's criticism of Bos' approach emanates from Ferraro's tendency to blur the distinction between A-track and B-track approaches. A further attempt to blur this distinction is found in Ferraro's "continuous leap" comment:

Eulerian infinitesimals ...when interpreted using the conceptual instruments available to modern mathematics, seem to be an *ambiguous* mixture of different elements, a *continuous leap* from a vague idea of *limit* to a *confused* notion of *infinitesimal*. (Ferraro 2004, 59, emphasis added)

Ferraro's comment appears in the "Conclusion" section in 2004. A virtually identical comment appears in the abstract in 2012, and yet again in the "Conclusion" section in Ferraro (2012, 24).

We argue however that Euler was far less "ambiguous" or "confused" than is often thought. Ferraro claims that when we allow our interpretation of Euler to be informed by modern mathematical concepts, we have no choice but to see Euler as fluidly moving from vague limits to confused infinitesimals. Let us now compare the interpretations by Ferraro and Laugwitz.

Ferraro's opposition to Laugwitz's interpretation is based on a conflation of ontology and practice (see Sect. 2.4). Laugwitz is not trying to read ontological foundations based on modern theories into Euler (which would indeed be "alien" to Euler's notions, to borrow Ferraro's terminology), but is rather focusing on Euler's mathematical practice.

Furthermore, Ferraro's own reading, with its emphasis on alleged continuity between limits and infinitesimals, is not sufficiently sensitive to the distinction between dual approaches (as analyzed by both Klein and Bos), which we refer to as A-track and B-track approaches.

Laugwitz's interpretation showed that drawing upon modern concepts allows us to see Euler's reasoning as clear and incisive. Indeed, we know since Robinson (1961) that Felix Klein's hunch concerning the *dual* approaches to the foundations of analysis in Klein (1932, 214) was right on target (see Sect. 4.3). In short, Ferraro assimilates two distinct approaches to the problem of the continuum without historical or mathematical evidence.

4.12 Rhetoric and Modern Interpretations

In his 2004 article (Ferraro 2004, 51, footnote 46) Ferraro sought to enlist the support of (Bos 1974, Appendix 2) for his (Ferraro's) opposition to interpreting Euler in terms of modern theories of infinitesimals. However, Henk Bos himself has recently distanced himself from the said Appendix 2 (part of his Doctoral thesis) in a letter sent in response to a question from one of the authors of the present text:

An interesting question, what made me reject a claim some 35 years ago? I reread the appendix and was surprised about the self assurance of my younger self. I'm less definite in my opinions today – or so I think. You're right that the appendix was not sympathetic to Robinson's view. Am I now more sympathetic? If you talk about "historical continuity" I have little problem to agree with you, given the fact that one can interpret continuity in historical developments in many ways; even revolutions can come to be seen as continuous developments. (Bos 2010)

The letter is reproduced with the author's permission. The shortcomings of Bos's Appendix 2 are analyzed in detail in Katz and Sherry (2013, section 11.3) and in Sect. 2.7 here. The clarification provided by Bos in 2010 weakens the claim of Bos's support for Ferraro's position on Robinson. Ferraro claims that

[Laugwitz and other] commentators use notions such as set, real numbers, continuum as a set of numbers or points, functions as pointwise relations between numbers, axiomatic method, which are modern, not Eulerian. (Ferraro 2004, 51)

Certainly, sets, real numbers, the punctiform continuum, and the modern notion of function are no Eulerian concepts. But has Laugwitz really committed the misdemeanors attributed to him by Ferraro? Ferraro does not provide any evidence for his claim, and one searches in vain the two articles Laugwitz (1989, 1992) cited by Ferraro for clues of such misdemeanors. On the contrary, Laugwitz warns the reader: "But one should have in mind that such concepts did not appear before set theory was established" (Laugwitz 1989, 242); and again:

Modern mathematicians should find of interest the fact that he [Cauchy] succeeded by using only very few concepts of an intensional quality, whereas we have become accustomed to using a great many extensional concepts based on set theory. (ibid.)

Laugwitz is clearly aware of the point that modern set theory is *alien* to Euler's ontology. However, as discussed in Sect. 2.4, Laugwitz is concerned with Euler's *procedures* rather than his ontology. Ferraro has surely committed a strawman fallacy in describing Laugwitz's scholarship as being "alien" to Euler.

To be sure, rhetorical and formal aspects of historical mathematics can be fruitfully studied in their own right. Yet an overemphasis on the rhetorical aspect to a point of dismissing as “extraneous” scholarly work that chooses to focus on the Eulerian mathematics *per se*, is untenable.

One may well wonder whether it sheds more light on Euler to observe, as Laugwitz does, that Euler’s infinitesimal procedures (Reeder’s *inferential moves*) turn out to depend on *hidden lemmas* (such as those concerning estimates for infinite sums of infinitesimals) but are otherwise remarkably robust and formalizable in modern infinitesimal mathematics; or whether it sheds more light to assert nonchalantly, as Ferraro does, that Euler considered infinitesimals to be exactly zero as a kind of *rhetorical device*, and that therefore there are neither “gaps” nor “hidden lemmas” in his proofs. Relating to Euler’s substitution

$$\frac{i-1}{i} = 1$$

as a *rhetorical device* as Ferraro does fails to explain why Euler sometimes disallows this type of substitution, as when Euler explains that

$$1 + \frac{x}{i}$$

cannot be replaced by 1 in factors of an infinite product in the passage from Euler (1748, §156) cited in Sect. 3.6. This passage from Euler explicitly contradicts Ferraro’s *rhetorical* reading.

4.13 Euler versus Berkeley, H. M. Edwards, and Gray

Cleric Berkeley’s critique tends to receive exaggerated attention in the literature. We second Fraser’s assessment to the effect that “Berkeley’s critique seems to have limited intrinsic merit” (Fraser 1999, 453, note 3). We now examine Ferraro’s approach to this critique. Ferraro states that

a [sic] unproblematic translation of certain chapters in the history of mathematics into modern terms tacitly assumes that the same logical and conceptual framework guiding work in modern mathematics also guided work in past mathematics. (Ferraro 2012, 2)

Here Ferraro expresses a legitimate concern. Certainly one shouldn’t project the conceptual framework guiding modern mathematics, upon an eighteenth century text. However, in the very next paragraph, Ferraro proceeds to state: “[Berkeley] did not cast any doubt upon the usefulness of the calculus in solving many problems of physics or geometry; nevertheless, he believed that it did not possess solid *foundations*” (ibid., emphasis added). Let us now examine the said foundations.

Berkeley’s “foundations,” if any are to be found, amount to an empiricist postulation of a minimal perceptual magnitude below which one cannot descend, and a consequent rejection of an infinitely divisible “extension” (i.e., continuum). This is clearly not the sense of the term *mathematical foundations* that Ferraro has in mind. Rather than being concerned with the latter, Berkeley voiced two separate criticisms: a *metaphysical* and a *logical* one; see Sherry (1987). The logical criticism concerns the alleged inconsistency expressed by the conjunction $(dx \neq 0) \wedge (dx = 0)$; see Sect. 3.3. The metaphysical

criticism is fueled by Berkeley's empiricist doubts about entities that are below any finite perceptual threshold.

Ferraro's description of Berkeley's criticism in terms of "foundations" falls prey to the very shortcoming he seeks to criticize, namely grafting modern concepts upon ones exploited in historical mathematics.

A related attempt by H. M. Edwards to sweep Euler's infinitesimals under an Archimedean rug in Edwards (2007) was analyzed in Kanovei et al. (2015). Edwards recently attempted to defend his comment that Euler's infinitesimal computations will not find a receptive audience today, when students are taught to shrink from differentials as from an *infectious disease* (Edwards 2015, 52, emphasis added). against our criticism in Kanovei et al. (2015). In recent years it has become popular to interpret differentials as 1-forms. This is fine, but it is not Euler's view, as we show in Kanovei et al. (2015) and in the present work. In his response to that article, Edwards clarifies that he does not dismiss Euler's use of differentials the way many others do. But it is not Edwards' *disposition* toward differentials that is the problem, but rather his *interpretation* of Euler's differentials. In his response, Edwards again fails to acknowledge that Euler's use of *bona fide* infinitesimals is not reducible to a purely algebraic algorithm.

Instead, Edwards indulges in rhetorical non-sequiturs against Robinson's framework, accusing it of being "far stranger than anything Euler could have imagined." Edwards further accuses the authors of Kanovei et al. (2015) of "entertain[ing] *strange ideas* about the concept of the infinite" (emphasis added). However, Edwards' remarks amount to a baseless ad hominem attack, since the article in question said not a word about either Robinson or his framework, focusing instead on the shortcomings of Edwards' take on Euler's work, including a forced constructivist paraphrase thereof and an anachronistic misattribution of the notion of *derivative* to Euler.

The book Edwards (1979) (unrelated) presents a sympathetic view of Robinson's framework, as does the book Tao (2014) which presents ultraproducts as a bridge between discrete and continuous analysis.

A year after the publication of H. Edwards' misguided analysis of Euler in Edwards (2007), J. Gray claimed that "Euler's attempts at explaining the foundations of calculus in terms of differentials, which are and are not zero, are *dreadfully weak*" (Gray 2008, 6). Prisoner of A-track methodology, Gray does not fail to succumb to Weierstrass's ghost when he claims in his *Plato's ghost* that Cauchy "defined what it is for a function ...to be continuous ...using careful, if not altogether unambiguous, *limiting arguments*" (Gray 2008a, 62, emphasis added). *Pace* Gray, it is inaccurate to claim that Cauchy defined continuity using *limiting arguments*. The word *limit* does appear in Cauchy's infinitesimal definition of continuity (reproduced only two pages later in *Plato's ghost*): "the function $f(x)$ is continuous with respect to x between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself" (Bradley and Sandifer 2009, 26). Evidently, *limits* do appear in Cauchy's definition (though they are replaced by *bounds* in Gray 2008a, 64). However, they appear only in the sense of the endpoints of the interval, rather than any sense related to the Weierstrassian notion of the limit.

Gray's grafting of Weierstrassian limits upon Cauchy's definition of continuity comes at a high price in anachronism. For a recent study of Cauchy based on Robinson's framework see Ciesielski and Miller (2016).

Ferraro could have used another term in place of *foundations*; however, the exaggerated significance attached to Berkeley's allegedly *foundational* critique becomes apparent when Ferraro declares that

The *crux of the question* lay in knowing what meaning to attribute to the equation $a + dx = a$. The exactness of mathematics required, according to Euler, that the differential dx should be precisely equal to 0: simply by assuming that $dx = 0$, the outrageous attacks on the calculus would be shown to lack any basis. (Ferraro 2012, 3, emphasis added)

Is this really the “crux of the question” as Ferraro contends? As discussed in Sect. 4.4, the *exact zero* infinitesimals are untenable and lead to insoluble paradoxes. Meanwhile, the answer to Berkeley's logical criticism lies elsewhere, namely the generalized notion of equality implied by both the Leibnizian *transcendental law of homogeneity* and the Eulerian *geometric comparison* (see Sect. 3.4) dubbed *the principle of cancellation* by Ferraro. Characterizing Berkeley's logical criticism as the “crux of the question” exaggerates the significance of his flawed empiricist critique of infinitesimals.

4.14 Aristotelian Continuum?

Euler defined *quantity* as that which could be increased or reduced in his *Elements of Algebra*: “Whatever is capable of increase or diminution, is called magnitude, or quantity” (Euler 1810, 1).

This may have been a common definition in Euler's time, but it was not the classical definition of quantity. What is called today the Archimedean axiom characterizes the ancient Greek notion of quantity, but it does not appear in modern mathematics until 1885 when it was rediscovered in Stolz (1885). Ferraro claims that

(1) Euler did not have the mathematical concept of set, nor the theory of real numbers nor the modern notion of function. (2) *He based the calculus on the classic notion of quantity.* (3) Quantity was conceived of as that which could be increased or reduced. (Ferraro 2012, 7, emphasis and numerals added)

Ferraro's first and last claims are beyond dispute, but his intermediate claim (italicized above) is dubious. Namely, the claim that Euler's notion of quantity was a “classic” one is unsupported by evidence. Ferraro seeks to connect Euler's quantity to the notion of quantity of unspecified *ancient Greeks* as well as to *the classical Aristotelian conception*: “[T]he Eulerian continuum is a slightly modified version of the Leibnizian continuum, as described by Breger [1992, 76–84], which, in turn, has many aspects in common with the classical Aristotelian conception” (Ferraro 2004, 37). Here Ferraro is referring to Breger (1992). Breger does write on page 76 that “Leibniz reprend la théorie aristotélicienne du continu” but in the same sentence he continues: “en y apportant trois modifications.” One of these modifications, according to Breger, is “l'emploi des grandeurs infinitésimales.” Ferraro's claim that Breger's description of the Leibnizian continuum has “many aspects in common” with the Aristotelian one appears to misrepresent Breger's position as far as infinitesimals are concerned.

Thus, while the Archimedean axiom belongs to the *classical* and *modern* notions of magnitude, it is found neither in *Euler's* characterisation of quantity as cited above, nor in Leibniz's view of quantity. See Sects. 3.8, 3.9, and 3.10 for a discussion of quantity from Euclid to Euler.

5 Conclusion

In his essay for the collection *Euler reconsidered*, Ferraro writes:

“Euler’s tripartite division of analysis was also the manifestation of his aim to reduce analysis as far as possible to algebraic notions; this latter term is used here to refer to notions deriving from an *infinitary extension of the principles of analysis of finite quantities*”. (Ferraro 2007, 45, emphasis added)

5.1 Cantor’s Ghost

Ferraro’s reference to Euler’s *infinitary extension of the principles of analysis of finite quantities* alludes to concepts such as infinite numbers and the associated infinite sums, or series, and infinite products. Infinite series and products are familiar syntactic features of modern, A-track, analysis as formalized by Cantor, Dedekind, and Weierstrass starting in the 1870s. We would like to comment on syntactic features that are noticeably absent from the said analysis. Cantor’s own position can be briefly summarized as follows:

Infinity, yes.

Infinitesimals, no.

In more detail, J. Dauben wrote:

Cantor devoted some of his most vituperative correspondence, as well as a portion of the *Beiträge*, to attacking what he described at one point as the ‘infinitesimal Cholera bacillus of mathematics’, which had spread from Germany through the work of Thomae, du Bois Reymond and Stolz, to infect Italian mathematics. (Dauben 1980, 216–217)

Dauben continues:

Any acceptance of infinitesimals necessarily meant that his own theory of number was incomplete. Thus to accept the work of Thomae, du Bois-Reymond, Stolz and Veronese was to deny the perfection of Cantor’s own creation. Understandably, Cantor launched a thorough campaign to discredit Veronese’s work in every way possible. (ibid.)

Ferraro elaborates on his *infinitary* comment cited above as follows: “*Euler was not entirely successful in achieving his aim, since he introduced infinitesimal considerations in various proofs*; however, algebraic analysis, as a particular field of mathematics, was clearly set out in the *Introductio*” (Ferraro 2007, 45, emphasis added). Given Ferraro’s acknowledgment that Euler exploits an *infinitary extension of the principles of analysis of finite quantities* as cited above, one might have expected that such an infinitary extension involves both infinite numbers *and* infinitesimals.

Yes Ferraro appears to feel, apparently following Cantor, that infinite series constitute legitimate and *successful* infinitary extensions, whereas inferences involving infinitesimals do *not*. However, infinite numbers i and infinitesimals ω in Euler are related by the simple equation

$$\omega = \frac{1}{i}$$

or more generally $i\omega = k$ where k is finite. Why would one be successful and the other *not entirely successful*? A possible source of the distinction is the reliance on a conceptual framework where infinite series admit suitable A-track proxies, whereas infinitesimals do not. Ferraro continues: “At the end of the eighteenth century, Euler’s plan to undertake an algebraic treatment of the broadest possible part of analysis of infinity had far-reaching consequences when Lagrange tried to reduce the whole of calculus to algebraic notions...” (Ferraro 2007, 45). Such a *broadest possible* algebraic framework is apparently not broad enough, in Ferraro’s view, to encompass Eulerian infinitesimals.

5.2 Primary Point of Reference?

In a similar vein, Fraser claims that

... classical analysis developed out of the older subject and it remains a *primary point of reference* for understanding the eighteenth-century theories. By contrast, non-standard analysis and other non-Archimedean versions of calculus emerged only fairly recently in somewhat *abstruse* mathematical settings that bear *little connection* to the historical developments one and a half, two or three centuries earlier. (Fraser 2015, 27, emphasis added)

For all his attempts to distance himself from Boyer’s idolisation of the triumvirate,¹² Fraser here commits himself to a position similar to Boyer’s. Namely, Fraser claims that modern punctiform A-track analysis is a *primary point of reference* for understanding the analysis of the past. His sentiment that modern punctiform B-track analysis *bears little connection to the historical developments* reveals insufficient attention to the procedure/ontology dichotomy. A sentiment of the inevitability of classical analysis is explicitly expressed by Fraser who feels that “classical analysis developed out of the older subject and it remains a primary point of reference for understanding the eighteenth-century theories” yet his very formulation involves circular reasoning. It is only if one takes classical analysis as a *primary point of reference* that it becomes plausible to conclude that it inevitably *developed out of the older subject*.

Such a position amounts to an unconditional adoption of the teleological *butterfly model* for the evolution of analysis, where infinitesimals are seen as an evolutionary dead-end. Elaborating an application of his butterfly/Latin dichotomy (see Sect. 2.6) to the case of infinitesimals, Ian Hacking writes:

If analysis had stuck to infinitesimals in the face of philosophical nay-sayers like Bishop Berkeley, analysis might have looked very different. Problems that were pressing late in the nineteenth century, and which moved Cantor and his colleagues, might have received a different emphasis, if any at all. This alternative mathematics might have seemed just as ‘successful’, just as ‘rich’, to its inventors as ours does to us. In that light, as Mancosu argued, transfinite set theory now looks much more like the result of one of Zeilberger’s random walks than an inevitable mathematical development. (Hacking 2014, 119)

¹² Historian Carl Boyer described Cantor, Dedekind, and Weierstrass as *the great triumvirate* in Boyer (1949, 298); the term serves as a humorous characterisation of both A-track scholars and their objects of adulation.

5.3 Paradigm Shift

Laugwitz’s pioneering articles from the 1980s such as Laugwitz (1987a, b, 1989) built upon earlier studies, particularly Robinson (1966) and Lakatos (1978). This work ushered in a new era in Euler and Cauchy scholarship. It became possible to dispense with, and go beyond, the worn clichés about unrigorous infinitesimalists and their inconsistent manipulations with mystical infinitesimals. In the case of Euler, it became possible to formalize and interpret some of his finest achievements in a way that sheds new light on the *methods* he used. This work points to a coherence of his formerly disparaged procedures based on the principle of cancellation, infinitesimals, and infinite numbers, and establishes a historical continuity in the procedures of infinitesimalists from Leibniz and Euler to Robinson.

Such a paradigm shift in Euler scholarship has encountered resistance from Ferraro, Fraser, Gray and other historians, who often cling to Procrustean (and often slavishly post-Weierstrassian) frameworks of Euler interpretation. Thus, Gray finds Euler’s explanations “dreadfully weak” but such a dismissive attitude toward Euler comes at a high price in anachronism when applied to the eighteenth century. Failing to distinguish clearly between procedural and ontological issues, these historians focus on the latter and stress the obvious point that modern set theory is alien to Euler’s ontology, thus falling back on strawman misrepresentations of the new wave of scholarship. The new scholarship accepts the obvious ontological point, and focuses rather on the methodological issues of the compatibility of Euler’s *inferential moves* and their proxies provided by procedures available in modern infinitesimal frameworks.

Seeing with what dexterity Leibniz and Euler operated on infinite sums as if they were finite sums, a modern scholar is faced with a stark choice. He can either declare that they didn’t know the difference between finite and infinite sums, or detect in their procedures a unifying principle (explicit in the case of Leibniz, and more implicit in the case of Euler) that, under suitable circumstances, allows one to operate on infinite sums as on finite sums. The former option is followed by Ferraro, and is arguably dictated by self-imposed limitations of an A-track interpretive framework. The latter option is the pioneering route of Robinson, Lakatos, Laugwitz, and others in interpreting the infinitesimal mathematics of Leibniz, Euler, and Cauchy.

Appendix: Analysis of Euler’s Proof

In Sect. 3.5 we summarized Euler’s derivation of the product decomposition for sine. The derivation of infinite product decompositions (9) and (10) as found in (Euler 1748, §156) can be broken up into seven steps as follows. Recall that Euler’s i is an infinite integer.

Step 1 Euler observes that

$$2 \sinh x = e^x - e^{-x} = \left(1 + \frac{x}{i}\right)^i - \left(1 - \frac{x}{i}\right)^i, \tag{23}$$

where i is an infinitely large natural number. To motivate the next step, note that the expression $x^i - 1 = (x - 1)(1 + x + x^2 + \dots + x^{i-1})$ can be factored further as a product $\prod_{k=0}^{i-1} (x - \zeta^k)$, where $\zeta = e^{2\pi\sqrt{-1}/i}$; conjugate factors can then be combined to yield a decomposition into real quadratic terms.

Step 2 Euler uses the fact that $a^i - b^i$ is the product of the factors

$$a^2 + b^2 - 2ab \cos \frac{2k\pi}{i}, \quad \text{where } 1 \leq k < \frac{i}{2}, \tag{24}$$

together with the factor $a - b$ and, if i is an even number, the factor $a + b$, as well.

Step 3 Setting $a = 1 + \frac{x}{i}$ and $b = 1 - \frac{x}{i}$ in (23), Euler transforms expression (24) into the form

$$2 + 2 \frac{x^2}{i^2} - 2 \left(1 - \frac{x^2}{i^2} \right) \cos \frac{2k\pi}{i}. \tag{25}$$

Step 4 Euler then replaces (25) by the expression

$$\frac{4k^2\pi^2}{i^2} \left(1 + \frac{x^2}{k^2\pi^2} - \frac{x^2}{i^2} \right), \tag{26}$$

justifying this step by means of the formula

$$\cos \frac{2k\pi}{i} = 1 - \frac{2k^2\pi^2}{i^2}. \tag{27}$$

Step 5 Next, Euler argues that the difference $e^x - e^{-x}$ is divisible by the expression

$$1 + \frac{x^2}{k^2\pi^2} - \frac{x^2}{i^2} \tag{28}$$

from (26), where “we omit the term $\frac{x^2}{i^2}$ since even when multiplied by i , it remains infinitely small” Euler (1988).

Step 6 As there is still a factor of $a - b = 2x/i$, Euler obtains the final equality (9), arguing that then “the resulting first term will be x ” (in order to conform to the Maclaurin series for $\sinh x$).

Step 7 Finally, formula (10) is obtained from (9) by means of the substitution $x \mapsto \sqrt{-1} x$.

Euler’s argument in favor of (9) and (10) was formalized in terms of a proof in Robinson’s framework in Luxemburg (1973). However, Luxemburg’s formalisation deviates from Euler’s argument beginning with steps 3 and 4, and thus circumvents the most problematic steps 5 and 6. A proof in Robinson’s framework, formalizing Euler’s argument step-by-step throughout, appeared in the article Kanovei (1988); see also McKinzie and Tuckey (1997) as well as the monograph (Kanovei and Reeken 2004, section 2.4a). This formalisation interprets problematic details of Euler’s argument on the basis of general principles in Robinson’s framework, as well as general analytic facts that were known in Euler’s time. Such principles and facts behind some early proofs exploiting infinitesimals are sometimes referred to as *hidden lemmas* in this context; see Laugwitz (1987a, 1989), McKinzie and Tuckey (1997).

For instance, a hidden lemma behind Step 4 asserts, on the basis of the evaluation of the remainder R of the Taylor expansion

$$\cos \frac{2k\pi}{i} = 1 - \frac{2k^2\pi^2}{i^2} + R,$$

that the quadratic polynomial $T_k(x) = 2 + 2 \frac{x^2}{i^2} - 2 \left(1 - \frac{x^2}{i^2} \right) \cos \frac{2k\pi}{i}$ as in (25) admits the representation

$$T_k(x) = C_k (U_k(x) + p_k \cdot x^2),$$

where C_k and p_k do not depend on x while

$$U_k(x) = 1 + \frac{x^2}{k^2\pi^2} - \frac{x^2}{i^2},$$

and for any *standard* real x and any finite or infinitely large integer $k \leq \frac{i}{2}$ the following holds:

- (1) if k is finite then p_k is infinitesimal, and
- (2) there is a real γ such that $|p_k| < \gamma \cdot k^{-2}$ for any infinitely large $k \leq \frac{i}{2}$.

This allows one to infer that the effect of the transformation of step 4 on the product of factors (25) is infinitesimal. See (Kanovei 1988, §4) as well as equation (11) on page 75 in Kanovei and Reeken (2004) for additional details.

Some *hidden lemmas* of a different kind, related to basic principles of nonstandard analysis, are discussed in McKinzie and Tuckey (1997, 43ff; see below).

What clearly stands out of Euler’s argument is his explicit use of infinitesimal quantities such as (25) and (26), as well as the approximate formula (27) which holds “up to” an infinitesimal of higher order. Thus, Euler exploited *bona fide* infinitesimals, rather than merely ratios thereof, in a routine fashion in some of his best work.

We now provide further technical details on a hyperreal interpretation of Euler’s proof of the product formula for the sine function. Our goal here is to indicate how Euler’s inferential moves find modern proxies in a hyperreal framework.

We discuss the *hidden lemmas* related to basic principles of nonstandard analysis following McKinzie and Tuckey (1997, 43ff), where it is argued that the Euler sine factorisation and similar constructions are best understood in the context of the following *hidden definition* in terms of modern nonstandard analysis. The following definition is borrowed from McKinzie and Tuckey (1997, 44).

Definition. A sum $a_1 + a_2 + a_3 + \dots$ is Euler-convergent (E-convergent) if and only if

- (i) a_k is defined by an elementary function,¹³
- (ii) for all infinite¹⁴ J , the sum $a_1 + a_2 + \dots + a_J$ is finite, and
- (iii) for all infinite pairs $J < K$, the sum $a_J + a_{J+1} + \dots + a_K$ is infinitesimal.

Similarly, a product $(1 + b_1)(1 + b_2)(1 + b_3)\dots$ is Euler-convergent if and only if (i) b_k is defined by an elementary function, (ii) for all infinite J , the product $(1 + b_1)(1 + b_2)\dots(1 + b_J)$ is finite, and (iii) for all infinitely large $J < K$, the product $(1 + b_J)(1 + b_{J+1})\dots(1 + b_K)$ differs infinitesimally from 1.

Next, McKinzie and Tuckey present a series of *hidden lemmas* implicit in Euler’s argument. The first such *hidden lemma* asserts that if the sums $a_1 + a_2 + \dots$ and $b_1 + b_2 + \dots$ are E-convergent and $a_k \simeq b_k$ (meaning that $a_k - b_k$ is infinitesimal) for all *finite* k , then

$$a_1 + a_2 + \dots + a_N \simeq b_1 + b_2 + \dots + b_N$$

¹³ The precise meaning of the modern term *elementary function* is discussed in McKinzie and Tuckey (1997, 43, footnote 23).

¹⁴ Here the terms *finite* and *infinite* correspond to *limited* and *infinitely large* in the terminology of McKinzie and Tuckey (1997).

for all N finite and infinite. To prove this lemma, it suffices to note that if $a_k \simeq b_k$ holds for all finite k , then, by Robinson's lemma (see e. g. Theorem 2.2.12, in Kanovei and Reeken 2004, 62), there is an infinite K such that $a_1 + \dots + a_k \simeq b_1 + \dots + b_k$ holds for all $k \leq K$.

The second *hidden lemma* asserts a similar property for products. The third *hidden lemma* asserts that if, for all finite x , the sums

$$f(x) = a_0 + a_1x + a_2x^2 + \dots \quad \text{and} \quad g(x) = b_0 + b_1x + b_2x^2 + \dots$$

are E-convergent and we have $[f(x) \simeq g(x)]$. This means that $a_0 + a_1x + a_2x^2 + \dots + a_Jx^J \simeq b_0 + b_1x + b_2x^2 + \dots + b_Kx^K$ for all infinite J, K . Note that the choice of J, K is immaterial by (ii) and (iii) of the definition of E-convergence. Then $a_n \simeq b_n$ for all n finite and infinite. A detailed analysis in McKinzie and Tuckey (1997) shows that these three lemmas, together with an additional sublemma, suffice to formalize Euler's derivations step-by-step in a hyperreal framework.

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