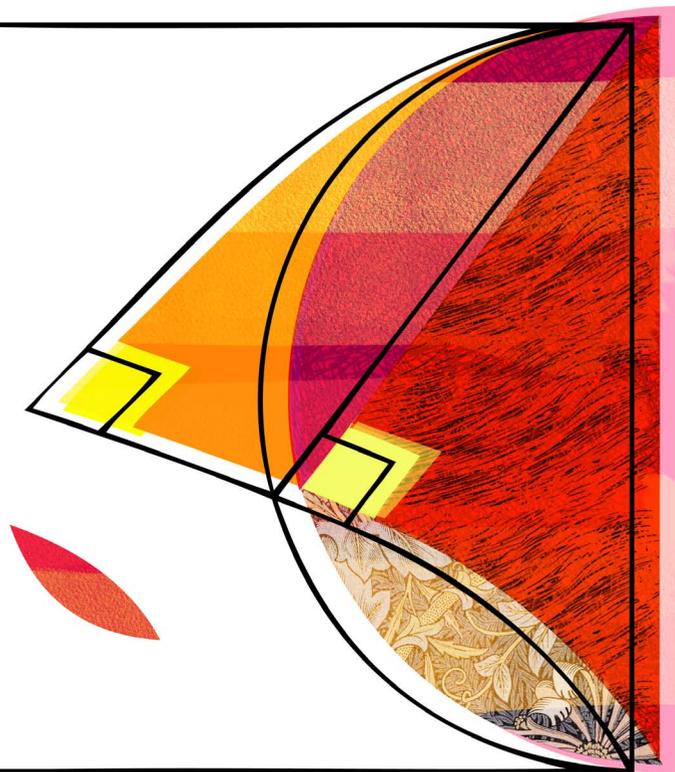


Piotr Błaszczuk · Anna Petiurenko



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# Introduction

**1** The reception of Euclid's *Elements* among philosophers and historians of mathematics in the 20th century was largely shaped by Thomas Heath's *Euclid. The Thirteen Books of the Elements*. His translation is based on Heiberg's edition of the Greek text. The first version appeared in 1908; the revised 1925 edition includes an extensive historical commentary. This revised version was reprinted by Dover Publications throughout the 20th century and is commonly cited as Heath (1956). Given the growing influence of the English language after the Second World War, most historians who do not specialize in Greek mathematics have come to know Euclid through this edition. To this day, *The Elements* are often cited in scholarly publications using Heath's translation.

In addition to detailed historical commentaries, Heath's edition includes improvements on Euclid's proofs – both original and historical – along with alternative demonstrations and other mathematically oriented remarks. One particularly famous contribution is the link he draws between Eudoxus's theory of proportion and Dedekind's theory of real numbers. This comparison functioned throughout the 20th century as a form of received wisdom and was reiterated even by renowned mathematicians such as John Conway.

The mathematical content of these commentaries does not form a coherent project; rather, it reflects the variety of interpretations of Euclid's geometry throughout the ages.

**2** A modern mathematician, if interested in elementary geometry, typically acquires it from Hilbert's *Grundlagen der Geometrie* (1899), and when referring to Euclid, does

so through the perspective of Hilbert's *Grundlagen*. As Hilbert's approach includes explicit references to the *Elements*, we view it as an interpretation of Euclid's geometry, and in this book, we present it from that perspective.

Other 20th-century systems of elementary geometry, such as Borsuk and Szmielew or Tarski and Szmielew, refer primarily to Hilbert's system, and we view them as an interpretation of Euclidean geometry rather than representations of Euclid's original system. Indeed, both systems are motivated by metamathematical questions. Borsuk and Szmielew adopt the continuity axiom (phrased in terms of Dedekind cuts) and demonstrate the categoricity of Euclidean geometry. Tarski's system, inspired by his results on the elimination of quantifiers in real closed fields, has been proven to be both complete and decidable.

Borsuk and Szmielew's system is considered as the foundation of geometry in Euclidean spaces  $\mathbb{R}^n$ , developed with techniques of modern linear algebra, topology, and analysis. On the other hand, the Tarski system was adopted by Naurboux and his colleagues to provide mechanical proofs of Euclid's propositions. In Chapter 3 of this book, we present the 2019 paper by Beeson, Narboux, and Wiedijk on mechanical proofs of Book I of the *Elements*.

Henri Poincaré, in his 1902 review of Hilbert's *Grundlagen*, wrote ironically about Hilbert's formalism: "One might put the axioms into a reasoning apparatus like the logic machine of Stanley Jevons, and see all the geometry out of it".<sup>1</sup>

In fact, it took a century to realize this project with respect to the *Elements*. Mechanical proof in geometry is a relatively new discipline, typically developed within the framework of algebra of polynomials. Mechanical proofs in synthetic geometry represent a small fraction of these endeavors.

**3** Robin Hartshorne (2000) completed Hilbert's project of interpreting Euclid within a modern framework and provided proofs for the propositions from Books I to IV based on Hilbert's axioms. However, Euclid's theory of proportion (Book V) and of similar figures (Book VI) is treated in an alternative manner. In Chapter 8 of our book, we discuss the reasons for this departure from Euclid, focusing on the challenges posed by Thales' Theorem (*Elements*, VI.2). In the same chapter, we also present

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<sup>1</sup>English translation adapted from Poincaré (1994, 150).

our mechanical proofs of Euclid's Book VI, which are, in a way, complementary to the results of Jean Narboux and his collaborators.

4 Ian Mueller was an eminent scholar emerging from the tradition of historians of Greek mathematics, who delved deeply into uncovering the relationships between ancient and modern geometry. In his 1981 book, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, he sought to compare Euclid's *Elements* with Hilbert's *Grundlagen*. As a characteristic feature of Hilbert's project, Mueller identifies a model-theoretic approach, which emphasizes the underlying structure as the object of interpretation through various models. Regarding his study, he writes: “[I]n this book, I will be contrasting the Greek use of the axiomatic method with the modern one and arguing that Greek mathematics should not be interpreted in terms of structure”.<sup>2</sup>

In addressing the foundations of geometry, our account of Book I of the *Elements* adopts a Hilbertian approach, particularly with respect to primitive concepts and relationships. We highlight the crucial differences between Euclid's geometry and modern geometry. Specifically, we show that in the *Elements*, the greater-than relation is taken as a primitive and applies to line segments, triangles, and angles. In contrast, in modern geometry – stemming from the Pasch tradition – this relation is introduced by definition.

According to this perspective, the same results may admit various demonstrations. Throughout Chapters 1 and 2 of our book, we emphasize the differences between Hilbert's and Euclid's systems by adopting a structural viewpoint. Moreover, the Euclidean tools – namely, the straightedge and compass – offer yet another lens through which to highlight the divergence between Euclid's and Hilbert's approaches.

In this context, we are guided by Hartshorne's theorem, which states that Cartesian planes over Pythagorean fields form models of Hilbert's system, while Cartesian planes over Euclidean fields form models of Euclid's system. A Euclidean field is an ordered field closed under the square root operation; indeed, extracting square roots is the algebraic analogue of straightedge and compass constructions.

Since every Pythagorean field is Euclidean, but not vice versa, this provides a justification for the claim that Hilbert's system does not offer a fully adequate or equivalent

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<sup>2</sup>See Mueller (1981, 10).

interpretation of Euclid's geometry.

**5** In the 1970s, a fierce exchange took place between mathematicians such as Van der Waerden, Weil, and Freudenthal, and a group of historians led by Sabetai Unguru, concerning the algebraic interpretation of Euclid's geometry – specifically Book II. In Chapter 4, we propose an intermediate position, grounded in the specific historical practices related to the use of the Pythagorean theorem. We show that, contrary to modern practice, Euclid does not transform the right-angled triangle formula  $c^2 = a^2 + b^2$  directly into  $a^2 = c^2 - b^2$ . Instead, he applies the concept of the gnomon to determine the value of  $a$  when the hypotenuse  $c$  and the leg  $b$  are given.

**6** As Book I of the *Elements* may be viewed as the study of the triangle, Book III is the study of the circle. Indeed, in Book III, Euclid addresses issues that are as fundamental as those in Book I. In Proposition III.2, he proves the distinction between an arc of a circle and a straight line. In Proposition III.16, he proves that a perpendicular to a radius at its endpoint is tangent to the circle – the most obvious and unquestioned claim throughout the history of mathematics, even though Euclid's proof is far from obvious. In Proposition III.35, he establishes another constant of his system: If a chord intersects a diameter, dividing the diameter into segments  $d_1$  and  $d_2$  and the chord into segments  $c_1$  and  $c_2$ , then the product  $d_1 \cdot d_2$  – expressed in Euclid's terms as a rectangle – equals  $c_1 \cdot c_2$ . This result holds for any other chord passing through the same intersection point.

Proposition III.35 provides the basis for a result which in modern mathematics is called the power of a point with respect to a circle. Euclid applies it to formulate a criterion for a line to be a tangent to a circle. Yet it might also be used to construct orthogonal circles, and indeed, a great part of a model of hyperbolic geometry known as the Poincaré disk. Thus, Book III provides the foundation for a model of non-Euclidean geometry. We discuss these topics in Chapter 5 of our book.

**7** Overall, our book offers various interpretations of Euclid's *Elements*, Books I to VI, both in technical and more informal terms.

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# Chapter 1

## Review of Book I

### 1. Absolute geometry. I.1–28

The term absolute geometry refers to propositions common to both Euclidean and hyperbolic geometry. From the perspective of the *Elements*, these are propositions that are proved without reference to the Parallel Postulate. Since the Parallel Postulate is invoked for the first time in Proposition I.29, the first 28 propositions of Book I belong to absolute geometry.

From the perspective of Hilbert's axioms, the situation appears slightly different, as they allow for different tools. Nevertheless, let absolute geometry be understood as a descriptive term referring to those theorems of Euclidean geometry that also hold true in hyperbolic geometry.

#### 1.1. Transportation of line segments. I.1–3

*1.1 To construct an equilateral triangle on the given line  $AB$ .*

Let  $a$  denote the line segment  $AB$ , and let point  $C$ , the third vertex of the desired triangle, be the intersection point of the circles  $(A, a)$  and  $(B, a)$ ; that is, circles centered at  $A$  and  $B$ , each with radius  $a$ .